INDEX TRANSFORMS

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Foreword

Integral transformations have traditionally become essential working tools for engineers and various other applied scientists. The Laplace transform, which undoubtedly is the most familiar and classical example of integral transformations, provides one of the basic (and most frequently used) tools in the solution of initial-value problems involving differential equations (and indeed also various families of Volterra integral equations of convolution type). The Fourier transform, while being suitable for solving boundary-value problems, is of fundamental importance in many areas of applied mathematics including, for example, the frequency spectrum analysis of time-varying wave forms. Although the aforementioned Laplace and Fourier transforms are by far the most widely (and effectively) used among all classical integral transforms, yet there are numerous other integral transformations which also have been used successfully in the solution of various boundary-value problems and in sundry other applications. One may include in this category such important integral transformations as the Mellin, Hankel, Stieltjes, Hilbert, Weierstrass, finite and discrete transforms.

The so-called **Index Transforms** are integral transformations whose kernels depend upon some of the indices (or parameters) of the special functions which are involved in them. The special functions associated with such integral transformations are, in general, of hypergeometric type. Indeed the Gaussian hypergeometric function:

$$ {}_2F_1(a,b;c;z) := 1 + \frac{a\cdot b}{c}\,\frac{z}{1!} + \frac{a(a+1)\cdot b(b+1)}{c(c+1)}\,\frac{z^2}{2!} + \cdots $$

$$ (a,b,c \in \mathbb{C}; \quad c \neq 0,-1,-2,\cdots; \quad z \in \mathcal{U} := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}) $$

and its familiar generalization, namely, the generalized hypergeometric function:

$$ \begin{aligned} {}_pF_q(\alpha_1,\cdots,\alpha_p;\ \beta_1,\cdots,\beta_q;\ z) & \\ \equiv {}_pF_q\left[\begin{matrix}\alpha_1,\cdots,\alpha_p; \\ \beta_1,\cdots,\beta_q;\end{matrix}\ z\right] & \\ := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\,\frac{z^n}{n!} & \end{aligned} $$

$$ (p \leq q+1; \ \alpha_j \in \mathbb{C}\,(j=1,\cdots,p); \ \beta_j \in \mathbb{C}\setminus\{0,-1,-2,\cdots\}\ (j=1,\cdots,q); $$

$$ p \leq q \text{ and } z \in \mathbb{C}; \ p = q+1 \text{ and } z \in \mathcal{U}; \ p = q+1,\, z \in \partial\mathcal{U}, \text{ and } \Re(\omega) > 0), $$

where

$$ \partial\mathcal{U} := \{z : z \in \mathbb{C} \text{ and } |z| = 1\} $$

and

$$\omega := \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j ,$$

and $(\lambda)_n$ denotes the **Pochhammer symbol** defined, in terms of Gamma functions, by

$$(\lambda)_n := \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n \in \mathbb{N} := \{1,2,3,\cdots\}), \end{cases}$$

have had a remarkably long and celebrated history. Furthermore, in Geometric Function Theory, which is the study of the relationship between the **analytic** properties of a given function $f(z)$ and the **geometric** properties of the image domain:

$$\mathcal{D} = f(\mathcal{U}),$$

it is an extremely difficult **open** problem to find a (useful) set of conditions on the coefficients a_n $(n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\})$ that are both *necessary* and *sufficient* for the function $f(z)$ *normalized* by

$$f(z) = z + \sum_{n=2}^{\infty} a_n\, z^n$$

to be in the class $\mathcal{S}$ of (normalized) analytic and univalent functions in the *open* unit disk $\mathcal{U}$. One of the several partial results in connection with this problem is provided by Louis de Branges' theorem of 1984, which asserts the truth of the **Milin conjecture** of 1971 and which implies the **Robertson conjecture** of 1936 and indeed also the famous **Bieberbach conjecture** of 1916:

$$f(z) \in \mathcal{S} \Rightarrow |a_n| \leq n \qquad (n = 2,3,4,\cdots),$$

where the equality holds true for all integers $n \geq 2$ only if

$$\begin{aligned} f(z) = \mathcal{K}_\phi(z) &:= \frac{z}{(1 - z e^{i\phi})^2} \\ &= \sum_{n=1}^{\infty} n\, e^{i(n-1)\phi}\, z^n \qquad (\phi \in \mathbb{R}), \end{aligned}$$

$\mathcal{K}_\phi(z)$ being a rotation of the Koebe function:

$$\mathcal{K}(z) \equiv \mathcal{K}_0(z) := \sum_{n=1}^{\infty} n\, z^n = \frac{z}{(1-z)^2}.$$

The key ingredients in de Branges' proof of the Milin conjecture, and hence also of the Robertson conjecture and the Bieberbach conjecture, include Löwner's differential equation and a certain nonnegativity result which may readily be put in the generalized hypergeometric form:

$$\frac{(\lambda+2)_n}{n!}\ {}_3F_2\left[\begin{matrix} -n, & \lambda+n+2, & \frac{1}{2}(\lambda+1); \\ & \lambda+1, & \frac{1}{2}(\lambda+3); \end{matrix}\ x\right] \geq 0$$

$$(0 \le x < 1; \quad \lambda \ge -2; \quad n \in \mathbb{N}_0).$$

It should be remarked in passing that the theory of special functions (and, especially, the generalized hypergeometric functions) has so far remained unavoidable in proving the aforementioned conjectures of far-reaching consequences in Geometric Function Theory. All these relatively recent developments in an area other, of course, than the so-called traditional areas of applications of generalized hypergeometric functions have naturally provided a new impetus for the study of the generalized hypergeometric functions, especially in connection with various subclasses of analytic functions. The present work: **Index Transforms**, on the other hand, aims at studying some of these special functions when their indices or parameters happen to be the vital components of the kernels $K(s,t)$ of the integral transforms under consideration.

Some of the main index transformations, which are considered in this book, include (i) the **Kontorovich-Lebedev transform** for which

$$K(s,t) = K_{is}(t) \qquad (s > 0)$$

in terms of the modified Bessel function of the third kind (popularly known as the Macdonald function); (ii) the **Mehler-Fock transform** for which

$$K(s,t) = \frac{\pi}{\cosh(\pi s)} \sqrt{\frac{\pi}{2}}\, P_{is-\frac{1}{2}}(t) \qquad (-\infty < s < \infty)$$

in terms of the familiar Legendre function; and (iii) the **Lebedev-Skalskaya transforms** for which

$$K(s,t) = \frac{2}{\pi} \cosh\left(\frac{\pi s}{2}\right) \left\{ \begin{matrix} \mathcal{R} \\ \mathcal{I} \end{matrix} \right\} K_{\frac{1}{2}+is}(t) \qquad (s > 0),$$

where, for convenience,

$$\left\{ \begin{matrix} \mathcal{R} \\ \mathcal{I} \end{matrix} \right\} K_{\frac{1}{2}+is} := \frac{K_{\frac{1}{2}+is} \{\pm\} K_{\frac{1}{2}-is}}{2 \left\{ \begin{matrix} 1 \\ i \end{matrix} \right\}},$$

again in terms of the Macdonald function. This book also systematically presents the L_p-theory of each of the aforementioned index transformations, and indeed also of numerous such generalizations of these index transformations as those whose kernels $K(s,t)$ involve the Gaussian and generalized hypergeometric functions, Meijer's G-function, and Fox's H-function. It should be recalled that the last two classes of higher transcendental functions (together, of course, with MacRobert's E-function) stemmed essentially from an attempt to give a meaning to the hypergeometric symbol ${}_pF_q$ when $p > q + 1$. And the H-function encompasses, as its special or limit cases, most (if not all) of the commonly used special functions of one variable and their numerous extensions and generalizations studied in the mathematical literature from time to time. Various *multivariable* generalizations of the H-function have also received considerable attention in recent years.

The following books are among those that have appeared recently on the subject of integral transforms and related topics:

1. Yu. A. Brychkov, H.-J. Glaeske, A.P. Prudnikov, and Vu Kim Tuan, ***Multidimensional Integral Transformations***, Gordon and Breach Science Publishers, Philadelphia, Reading, Paris, Montreux, Tokyo, and Melbourne, **1992**;

2. Nguyen Thanh Hai and S.B. Yakubovich, ***The Double Mellin-Barnes Type Integrals and Their Applications to Convolution Theory***, World Scientific Publishing Company, Singapore, New Jersey, London, and Hong Kong, **1992**;

3. H.M. Srivastava and R.G. Buschman, ***Theory and Applications of Convolution Integral Equations***, Mathematics and Its Applications **79**, Kluwer Academic Publishers, Dordrecht, Boston, and London, **1992**.

4. Semen B. Yakubovich and Yurii F. Luchko, ***The Hypergeometric Approach to Integral Transforms and Convolutions***, Mathematics and Its Applications **287**, Kluwer Academic Publishers, Dordrecht, Boston, and London, **1994**.

Two of these works were co-authored by Dr. Semen B. Yakubovich himself; Srivastava and Buschman (1992) consider essentially convolution integral equations with special function kernes; and Brychkov *et al.* (1992) deal mainly with the **multidimensional** analogues and extensions of integral transformations. The aforementioned work of Yakubovich and Luchko (1994) is undoubtedly most relevant to the subject-matter of the present book. Each of these recent works [and, especially, Yakubovich and Luchko (1994)] has been found to be a useful addition to the existing literature on integral transformations and related topics. And it is my sincere hope that **Index Transforms** will prove to be at least as successful as its predecessors listed above.

In conclusion, I am happy to recommend this **state-of-the-art** presentation of index transformations to all those graduate students and researchers (and other users of mathematics) in the fields of mathematical, physical, astrophysical, statistical, and engineering sciences who may find the various mathematical tools developed in **Index Transforms** to be potentially applicable in their works.

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June 1995

Contents

Preface

This book is intended as an attempt to give a systematic investigation of the integral transforms whose kernels depend from the index or parameter of special functions. At first as is known the most familiar transforms are the Fourier, the Laplace and the Mellin integral transforms. However, as is shown in this volume these operators by their composition properties can generate other integral transforms. In particular, compositions with simple arguments generated the Mellin and the Laplace convolution type transforms (the Hankel, Stieltjes, Hilbert, Weierstrass, sine and cosine Fourier transforms, Meijer transform etc.) We refer the reader on this matter to the wide list of citations in the bibliography.

The key transforms considered here are the Kontorovich-Lebedev and the Mehler-Fock integral transforms whose kernels are correspondingly either the modified Bessel function of the third kind of the pure imaginary index (the Macdonald function) or the associated Legendre function of the first kind. As it was established these transforms involve compositions of the above mentioned operators of convolution type with functional argument in general case. Furthermore, a special kind of integration is realized within the inversion formulae for the Kontorovich-Lebedev and the Mehler-Fock transforms. Precisely speaking it contains the integration with respect to an index of the special functions noted above. Such integral operators are known as index transforms and the L_p-theory of theirs has been developed in this book. Considering special functions of hypergeometric type whose Mellin transform is the ratio of products of Euler's gamma-functions we generalized the Kontorovich-Lebedev and the Mehler-Fock transforms for other kernels. Thus we obtained the Olevskii ${}_2F_1$-index transform with Gauss's hypergeometric function as the kernel, the Lebedev-Skalskaya index transform with either a real or an imaginary part of an arbitrary complex index of the Macdonald function, Lebedev's index transform with the square of the Macdonald function, the Wimp-Yakubovich index transform by index of Meijer's G-function and Fox's H-function. The hypergeometric structure of special functions considered in this book and the Mellin transform technique allows us to introduce index transforms with arbitrary kernels like Watson transforms concerning Mellin convolution type operators. These objects are discussed in detail in the present volume.

The organization of materials is as follows. In Chapter 1 we give some preliminary notions of the L_p-theory of the Lebesgue integral and various properties of the hypergeometric type special functions. Also included are the important Fubini and Lebesgue theorems, the weighted Hölder inequalities, elements of the theory of the Mellin-Barnes integrals and the Slater theorem, Meijer's G-function and Fox's

H-function and their particular cases. We completed the material of Chapter 1 by asymptotic behavior of special functions as well as by argument and by index. Elements of the theory of the Mellin convolution type transforms are given too.

Chapter 2 deals with the L_p-properties of the Kontorovich-Lebedev transform. The mapping properties in the weighted Lebesgue spaces and inversion formula are established. Special case of the Hilbert space is considered and the Parseval equality is obtained. With the aid of familiar Laplace convolution the respective convolution Hilbert space is introduced, and another method of inversion of the Kontorovich-Lebedev transform is demonstrated. Finally, the index-convolution Kontorovich-Lebedev operator is discussed as mapping from the Lebesgue space of functions of one variable into the two-dimensional Lebesgue weighted L_p-space.

Similar questions are considered for the Mehler-Fock index transform in Chapter 3. We give various methods of its inversion including the use of the composition representation of the Mehler-Fock transform by means of the Kontorovich-Lebedev transform and the Hankel transform.

Very important objects are exhibited in Chapter 4. We introduced so-called convolution of the Kontorovich-Lebedev transform that contains the double integral with the symmetric exponential kernel and has remarkable operational properties. Several useful estimates are proved for this convolution in power-weighted $L_{\nu,p}$-spaces. The corresponding Parseval equality and the factorization property are established. The convolution Hilbert space is considered and applied for inversion of the Kontorovich-Lebedev transform. Some integral equations of convolution type are demonstrated and their solutions are discussed by using the theory of Banach algebras in commutative normed rings.

In Chapter 5, the extension of the Kontorovich-Lebedev and the Mehler-Fock transforms on the complex domain is given. Furthermore, the identifications of images in the analogs of the Bergman-Selberg and Szegö Hilbert spaces are presented with using the theory of reproducing kernels. In addition, the analogs of the Paley-Wiener theorem for the Kontorovich-Lebedev type index transforms are established. More general L_p-cases of the Hardy spaces are considered too. This chapter contains general index transforms that are constructed and inverted by means of their Mellin-Barnes integral representations and using the Mellin transform L_p-theory. As examples some known and new pairs of the index transforms are presented. This material includes composition theorems for these transforms and their inversions. The corresponding kernels are explicitly calculated in terms of hypergeometric functions.

The series of examples of the index transforms is continued in Chapters 6–7. We considered so-called Lebedev-Skalskaya type operators and comprised some cases of index transforms with hypergeometric functions of ${}_pF_q$-type as well as kernels that involve combinations of the cylindrical functions.

Furthermore, in the final chapter we announce that our composition approach can be spread on the essentially multidimensional Kontorovich-Lebedev transform. This index transform one can represent as a composition of the multidimensional Fourier transform and some modification of the multidimensional Laplace transform.

For the sake of convenience the author index, the subject index and notations are given at the end of the book.

This book is written primarily for researchers and graduate students in the areas of special functions and integral transforms. Research workers and other users of integral transforms, special functions, convolutions and integral equations shall find here useful information to use in applications.

This project was conceived and realized during my time as a research fellow in the Department of Applied Mathematics of Fukuoka University, Japan. At the time, I was on sabbatical leave from the Byelorussian State University of Minsk, Belarus.

I wish to express my deep thanks to Professor Megumi Saigo for his kind invitation, hospitality and cooperation at Fukuoka University. I am grateful to the staff of the Department of Applied Mathematics that provided me such excellent research and computer facilities. Also, sincere appreciation is offered to Richard Oberc of the Department of Humanities of Fukuoka University for his assistance in the English language and moral support.

I wish to thank the World Scientific Publishing Co. for the opportunity to publish this Work. Especially, I am pleased to thank the editor of this book Ms. Dr. Anju Goel for reading the manuscript and for suggesting a number of necessary and invaluable improvements.

Finally, I would like to note that some chapters of this book are based on my lectures given for graduate students at the Department of Applied Mathematics at Fukuoka University. The series of recent results were announced at the '94 bi-annual meetings of the Mathematical Society of Japan.

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March 1995

Chapter 1

Preliminaries

This chapter is very important for our considerations throughout this book. We present various notions and statements known in mathematical analysis, namely in topics of hypergeometric type functions and Mellin convolution type integrals which are repeatedly used to construct the index transforms theory.

1.1 The spaces L_p and $L_p(\rho)$

We assume that the reader is familiar with Lebesgue measurability of functions and the Lebesgue integral. Let $\Omega = [a, b]$, $-\infty \leq a < b \leq \infty$. We denote by $L_p = L_p(\Omega)$ the set of all Lebesgue measurable functions $f(x)$, complex valued in general for which $\int_\Omega |f(x)|^p dx < \infty$, where $1 \leq p < \infty$. In case $\Omega = [-\infty, \infty]$ we shall denote it by $\mathbf{R}$ and for $\Omega = [0, \infty]$ let us put $\Omega = \mathbf{R}_+$. We set

$$||f||_{L_p(\Omega)} = \left(\int_\Omega |f(x)|^p dx \right)^{1/p}. \tag{1.1}$$

If $p = \infty$ the space $L_p(\Omega)$ is defined as the set of all measurable functions with a finite norm

$$||f||_{L_\infty(\Omega)} = \operatorname*{ess\,sup}_{x \in \Omega} |f(x)|, \tag{1.2}$$

where $\operatorname{ess\,sup} |f(x)|$ is an essential maximum of the function $|f(x)|$-see details in Kolmogorov and Fomin [1].

Everywhere below assume that $1 \leq p \leq \infty$. As usual two equivalent functions, i.e. differing on a set of zero measure, are considered to be equal to one element of the space $L_p(\Omega)$. That is, they are not distinguished as elements of this space.

For norms (1.1) and (1.2) we shall also use notations

$$||f||_p = ||f||_{L_p} = ||f||_{L_p(\Omega)}. \tag{1.3}$$

Sometimes we shall use the notation $L_p(\gamma - i\infty, \gamma + i\infty)$, when the set Ω is a vertical straight line at complex plain s, $\Re s = \gamma$, where by sign "$\Re s$" we denote a real part of complex variable s.

We use very often the notation $\int_\Omega f(x)dx$ to denote that the integral exists in some sense or other. For instance, we mean the Lebesgue integral of $f(x)$ over Ω in the strict sense, implying that the integral is absolutely convergent, i.e $\int_\Omega |f(x)|dx < \infty$. If for example, $f(x)$ is integrable over $[1/E, E]$ for every $E > 0$ that we denote by

$$\int_0^\infty f(x)dx = \lim_{E\to\infty} \int_{1/E}^E f(x)dx, \tag{1.4}$$

meaning that limit (1.4) exists and we call such integral as improper one.

By

$$\text{l.i.m.}_{E\to\infty} \int_{1/E}^E f(x,t)dt \tag{1.5}$$

(limit in mean or L_p-sense) we denote a function $\varphi(x)$ such that

$$\lim_{E\to\infty} \left\| \varphi - \int_{1/E}^E f(x,t)dt \right\|_p$$

$$= \lim_{E\to\infty} \left(\int_\Omega \left| \varphi(x) - \int_{1/E}^E f(x,t)dt \right|^p dx \right)^{1/p} = 0. \tag{1.6}$$

Let us give some properties of the spaces L_p:

a) *The Minkowski inequality*

$$||f+g||_{L_p(\Omega)} \leq ||f||_{L_p(\Omega)} + ||g||_{L_p(\Omega)}, \tag{1.7}$$

so that $L_p(\Omega)$ is a normed space. It is also known that $L_p(\Omega)$ is a complete space;

b) *The Hölder inequality*

$$\int_\Omega |f(x)g(x)|dx \leq ||f||_{L_p(\Omega)} ||g||_{L_q(\Omega)}, \quad q = p/(p-1), \tag{1.8}$$

where $f(x) \in L_p(\Omega)$, $g(x) \in L_q(\Omega)$. Index q, which is connected with p by the relation

$$\frac{1}{p} + \frac{1}{q} = 1, \tag{1.9}$$

is called *conjugate* to p. We note that (1.8) is true if $1 \leq p \leq \infty (q = \infty$, if $p = 1$, and $q = 1$, if $p = \infty)$;

c) *the Fubini theorem* which allows us to interchange the order of integration in iterated integrals:

Theorem 1.1. *Let $\Omega_1 = [a,b], \Omega_2 = [c,d], -\infty \leq a < b \leq \infty,\ -\infty \leq c < d \leq \infty$, and let $f(x,y)$ be a measurable function defined on $\Omega_1 \times \Omega_2$. If at least one of the integrals*

$$\int_{\Omega_1} dx \int_{\Omega_2} f(x,y)dy, \quad \int_{\Omega_2} dy \int_{\Omega_1} f(x,y)dx, \quad \int\int_{\Omega_1\times\Omega_2} f(x,y)dxdy,$$

is absolutely convergent then they coincide.

The generalized *Minkowski inequality*

$$\left(\int_{\Omega_1} dx \left|\int_{\Omega_2} f(x,y)dy\right|^p\right)^{1/p} \le \int_{\Omega_2} dy \left(\int_{\Omega_1} |f(x,y)|^p dx\right)^{1/p}, \qquad (1.10)$$

adjoining the Fubini theorem is also true;

d) the property of mean continuity for functions in L_p.

Lemma 1.1. *Let $f(x) \in L_p(\Omega), 1 \le p < \infty$. Then*

$$\int_\Omega |f(x+h) - f(x)|^p dx \to 0 \qquad (1.11)$$

as $h \to 0$, the function $f(x)$ is continued by zero for $x + h \notin \Omega$;

e) let $C_0^\infty(\Omega)$ be the space of all infinitely differentiable functions finite on Ω. Finiteness on Ω means that $f(x) \equiv 0$ in the neighborhood of the end-points $x = a$ and $x = b$ of the set $\Omega = [a,b], -\infty \le a < b \le \infty$. The space $C_0^\infty(\Omega)$ is dense in $L_p(\Omega)$, $1 \le p < \infty$;

f) the so-called *Lebesgue dominated convergence theorem* on passage to a limit under the integral sign:

Theorem 1.2. *Let the function $f(x,h)$ have summable majorant: $|f(x,h)| \le F(x)$, where $F(x)$ does not depend on the parameter h and $F(x) \in L_1(\Omega)$. If $\lim_{h\to 0} f(x,h)$ exists for almost all x, then*

$$\lim_{h\to 0} \int_\Omega f(x,h)dx = \int_\Omega \lim_{h\to 0} f(x,h)dx. \qquad (1.12)$$

The proof of the above properties can be found for example in the book by Kolmogorov and Fomin [1]. We shall use also the following statement.

Theorem 1.3. *Let $K(t) \in L_1(\mathbf{R})$ and $\int_\mathbf{R} K(t)dt = 1$. Then the averaging*

$$\int_{-\infty}^{\infty} K(t) f(x - \varepsilon t)dt = \frac{1}{\varepsilon}\int_{-\infty}^{\infty} K\left(\frac{t}{\varepsilon}\right) f(x-t)dt \qquad (1.13)$$

of the function $f(x) \in L_p(\mathbf{R})$, $1 \le p < \infty$, converges to $f(x)$ as $\varepsilon \to 0$ in $L_p(\mathbf{R})$-norm. Moreover, if $|K(t)| \le A(|t|)$, where $A(r) \in L_1(\mathbf{R}_+)$, and monotonically decreases then averaging (1.13) *converges to $f(x)$ almost everywhere.*

The proof of this theorem see, for example, in the book by Bochner [1]. In particular, the known *Poisson kernel*

$$P(t) = \frac{1}{\pi}\frac{1}{t^2+1} \qquad (1.14)$$

satisfies the above theorem and moreover it is not difficult to establish the following analog of Theorem 1.3 which is also worth mentioning.

Theorem 1.4. *Let the function* $M(x,t,\varepsilon)$ *be bounded uniformly as the function of three variables* $x > 0, t \in \mathbf{R}, \varepsilon > 0$ *and* $\lim_{\varepsilon \to 0+} M(x,t,\varepsilon) = 1$. *Then the averaging*

$$I_\varepsilon(x) = \frac{1}{\pi} \int_{-1/\varepsilon}^{\infty} \frac{M(x,t,\varepsilon)(1+\varepsilon t)}{t^2+1} f(x(1+\varepsilon t))dt \tag{1.15}$$

of the function $f(x) \in L_p(\mathbf{R}_+)$, $1 \le p < \infty$, *converges to* $f(x)$ *as* $\varepsilon \to 0$ *in* $L_p(\mathbf{R}_+)$-*norm. In addition, averaging* (1.15) *converges to* $f(x)$ *almost everywhere.*

Proof. Indeed, invoking with generalized Minkowski inequality (1.10) and conditions of this theorem we have the estimate

$$||I_\varepsilon(x)||_{L_p(\mathbf{R}_+)} \le C \int_{-1/\varepsilon}^{\infty} \frac{(1+\varepsilon t)}{t^2+1} ||f(x(1+\varepsilon t))||_{L_p(\mathbf{R}_+)} dt$$

$$\le C||f||_{L_p(\mathbf{R}_+)} \int_{-\infty}^{\infty} \frac{(1+|t|)^{1-1/p}}{t^2+1} dt < \infty, \tag{1.16}$$

where $C > 0$ is an absolute constant and the last integral is convergent due to the range of exponent p. Hence in view of Lebesgue's dominated Theorem 1.2 we obtain

$$||f(x) - I_\varepsilon(x)||_{L_p(\mathbf{R}_+)} = \frac{1}{\pi} \Bigg|\Bigg| \int_{-\infty}^{\infty} \frac{dt}{t^2+1}$$

$$\times [f(x) - M(x,t,\varepsilon)(1+\varepsilon t)H(1+\varepsilon t)f(x(1+\varepsilon t))]||_{L_p(\mathbf{R}_+)}$$

$$\le \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dt}{t^2+1} ||f(x)$$

$$-M(x,t,\varepsilon)(1+\varepsilon t)H(1+\varepsilon t)f(x(1+\varepsilon t))||_{L_p(\mathbf{R}_+)} \to 0+, \quad \varepsilon \to 0+. \tag{1.17}$$

Here by $H(t)$ we denoted the Heaviside function $H(t) = 1,\ t \ge 0,\ H(t) = 0,\ t < 0$. Convergence almost everywhere follows from Theorem 1.3 by properties of the Poisson kernel (1.14). Theorem 1.4 is proved. •

Let us define now the weighted L_p-space $L_p(\rho)$.

Definition 1.1. *Let* $\rho(x)$ *be a nonnegative function. Denote by* $L_p(\rho) = L_p(\Omega;\rho)$ *the space of functions* $f(x)$, *measurable on* Ω *for which*

$$||f||_{L_p(\rho)} = \left(\int_\Omega \rho(x)|f(x)|^p dx\right)^{1/p} < \infty. \tag{1.18}$$

We shall deal below on the whole with power-exponential weights. Especially note here the case when $\rho(x) = x^{\nu p-1}$, $x \in \mathbf{R}_+, \nu \in \mathbf{R}$. This weighted L_p-space we shall denote by $L_{\nu,p}(\mathbf{R}_+)$ with norm

$$||f||_{L_{\nu,p}(\mathbf{R}_+)} = \left(\int_0^\infty |x^\nu f(x)|^p \frac{dx}{x}\right)^{1/p} < \infty. \tag{1.19}$$

The space $L_p(\rho)$ is a Banach one in view of the isometry

$$||f||_{L_p(\rho)} = ||\rho^{1/p} f||_{L_p(\Omega)}. \tag{1.20}$$

From (1.8) owing to (1.20) an analog of the Hölder inequality for weighted spaces follows

$$\int_\Omega |f(x)g(x)|dx \leq ||f||_{L_p(\rho)} ||g||_{L_q(\rho^{1-q})}, \quad 1 < p < \infty. \tag{1.21}$$

Lastly we give the following Banach theorem.

Theorem 1.5. *Let A and B be linear bounded operators in a Banach space X. If $A\varphi \equiv B\varphi$ for φ in the set which is dense in X, then $A\varphi \equiv B\varphi$ for all $\varphi \in X$.*

1.2 Special functions of the hypergeometric type

Here we give elements of theory of special functions of the hypergeometric type, mentioning various integral and series representations and asymptotic properties. More detailed information can be found in the books by Erdélyi et al. [1], Marichev [1], Prudnikov et al. [1-5] and Olver [1].

A. **Gamma-function** $\Gamma(z)$. The Euler integral of the second kind

$$\Gamma(z) = \int_0^\infty e^{-x} x^{z-1} dx, \ \Re z > 0 \tag{1.22}$$

is called the gamma-function. It is obviously absolutely convergent for all $z \in \mathbf{C}$ for which $\Re z > 0$ and uniformly convergent by $t \in \mathbf{R}, z = \Re z + it$. Here $x^{z-1} = e^{(z-1)\log x}$. From the relation (1.22) one can derive the fact that the gamma-function is an analytic function in the half-plane $\Re z > 0$ (see Erdélyi et al. [1]). The gamma-function is extended to the half-plane $\Re z \leq 0$, $z \neq 0, -1, -2, \ldots$ by analytic continuation of this integral. Namely, the reduction formula

$$\Gamma(z+1) = z\Gamma(z), \ \Re z > 0, \tag{1.23}$$

obtained from relation (1.22) by integration by parts, yields the equality

$$\Gamma(z) = \frac{\Gamma(z+n)}{z(z+1)(z+2)\dots(z+n-1)}, \tag{1.24}$$

$$\Re z > -n, \; n = 1, 2, \dots, \; z \neq 0, -1, \dots,$$

which allows to carry out the analytic continuation into the half-plane $\Re z > -n$ for any n. The other method of analytic continuation is based on the Euler-Gauss formula

$$\Gamma(z) = \lim_{n\to\infty} \frac{n! n^z}{z(z+1)\dots(z+n)}, \; z \neq 0, -1, -2, \dots, \tag{1.25}$$

which can be obtained from relation (1.22). The following useful estimate

$$|\Gamma(z)| \leq |\Gamma(\Re z)| \tag{1.26}$$

is a consequence of formula (1.25) (see Olver [1]). It follows from relation (1.24) that the function $\Gamma(z)$ is analytic everywhere in the complex plane except $z = 0, -1, -2, \dots$, where it has simple poles and is represented by the formula

$$\Gamma(z) = \frac{(-1)^k}{k!(z+k)} [1 + O(z+k)], \; z \to -k, \; k = 0, 1, 2, \dots. \tag{1.27}$$

Here and everywhere in the book, the equality $f(z) = O(g(z))$, $z \to a$ means $|\frac{f(z)}{g(z)}| < M < \infty$ as $|z-a| < \varepsilon$. The relation $f(z) = o(g(z))$, $z \to a$ means that $\lim\limits_{z\to a} \frac{f(z)}{g(z)} = 0$ and the equivalence $f(z) \approx g(z)$, $z \to a$ means that $\lim\limits_{z\to a} \frac{f(z)}{g(z)} = 1$. From representation (1.27), we have

$$\text{res}_{z=-k}\Gamma(z) = \frac{(-1)^k}{k!}, \; k = 0, 1, 2, \dots. \tag{1.28}$$

We formulate some other properties of the gamma-function now:

a) supplement formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}; \tag{1.29}$$

b) Gauss-Legendre formula

$$\Gamma(nz) = \frac{n^{nz-\frac{1}{2}}}{(2\pi)^{\frac{(n-1)}{2}}} \prod_{k=0}^{n-1} \Gamma\left(z + \frac{k}{n}\right), \; n = 2, 3, \dots; \tag{1.30}$$

c) Weierstrass formula

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{k=1}^{\infty} \left(\left(1 + \frac{z}{k}\right) e^{-z/k} \right), \tag{1.31}$$

where $\gamma = \lim\limits_{n\to\infty} \left(\sum\limits_{m=1}^{n} \frac{1}{m} - \log n \right)$ is Euler's constant;

d) asymptotic Stirling's formula

$$\Gamma(z) = \sqrt{2\pi} z^{z-\frac{1}{2}} e^{-z} \left[1 + O(1/z)\right], \ |\arg(z)| < \pi, \ |z| \to \infty \tag{1.32}$$

and its corollary

$$|\Gamma(x+iy)| = \sqrt{2\pi} |y|^{x-\frac{1}{2}} e^{-\pi|y|/2} \left[1 + O(1/y)\right], \ |y| \to \infty, \tag{1.33}$$

which related to the *first formula of Binet*

$$\log \Gamma(z) = (z - 1/2) \log z - z + \frac{1}{2} \log(2\pi)$$

$$- \int_0^\infty \left(\frac{1}{2} + \frac{1}{t} - \frac{1}{1-e^{-t}}\right) \frac{e^{-zt}}{t} dt, \quad \Re z > 0. \tag{1.34}$$

We introduce here for further convenience Slater's notation (see Slater [1], Marichev [1])

$$\Gamma \begin{bmatrix} \alpha_1, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_q \end{bmatrix} = \frac{\Gamma(\alpha_1) \ldots \Gamma(\alpha_p)}{\Gamma(\beta_1) \ldots \Gamma(\beta_q)}. \tag{1.35}$$

B. **Pochhammer symbol** $(z)_n$ with integer n is defined by the equality

$$(z)_n = \frac{\Gamma(z+n)}{\Gamma(z)} = \prod_{k=0}^{n-1} (z+k). \tag{1.36}$$

From relation (1.36) and properties of gamma-function we obtain the following formulae

$$(1)_n = n!, \tag{1.37}$$

$$(1-n-z)_n = (-1)^n (z)_n, \tag{1.38}$$

$$(z)_{-n} = \frac{(-1)^n}{(1-z)_n}, \tag{1.39}$$

$$(z)_{2n} = 4^n (z/2)_n ((1+z)/2)_n, \tag{1.40}$$

$$\Gamma(z-n) = \frac{(-1)^n}{(1-z)_n} \Gamma(z). \tag{1.41}$$

C. **Euler integral of the first kind**

$$B(s,t) = \int_0^1 x^{s-1} (1-x)^{t-1} dx, \ \Re s > 0, \ \Re t > 0 \tag{1.42}$$

is called the beta-function. It is related to the gamma-function by the formula

$$B(s,t) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}. \tag{1.42$'$}$$

Using representation (1.42)$'$, obtain the following useful relations

$$B(s,t) = \int_0^\infty \frac{x^{s-1}}{(1+x)^{s+t}} dx, \ \Re s > 0, \ \Re t > 0, \tag{1.43}$$

$$|B(s,t)| \leq B(\Re s, \Re t), \ \Re s > 0, \ \Re t > 0. \tag{1.44}$$

D. **The generalized hypergeometric function** ${}_pF_q(z)$ is defined as a sum of the series

$$\begin{aligned} {}_pF_q\left[(a)_p;\ (b)_q;\ z\right] &\equiv {}_pF_q\left[\begin{matrix}(a)_p; \\ (b)_q;\end{matrix} z\right] \equiv {}_pF_q\left[\begin{matrix}a_1,\ldots,a_p; \\ b_1,\ldots,b_q;\end{matrix} z\right] \\ &\equiv \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{j=1}^{q}(b_j)_n} \frac{z^n}{n!}. \end{aligned} \tag{1.45}$$

The series on the right-hand side of relation (1.45) is absolutely convergent for all values of z, both real and complex, when $p \leq q$. Further, when $p = q+1$, the series converges if $|z| < 1$. It converges when $z = 1$ if

$$\Re\left[\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j\right] > 0,$$

and when $|z| = 1, \ z \neq 1$, if

$$\Re\left[\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j\right] > -1.$$

For other values of z the generalized hypergeometric function is defined as an analytic continuation of this series. One of the methods of such a continuation is the Mellin-Barnes integral representation

$$\begin{aligned} &{}_{q+1}F_q\left[(a)_{q+1};\ (b)_q;\ z\right] \\ &= \frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{j=1}^{q+1}\Gamma(a_j)} \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\prod_{j=1}^{q+1}\Gamma(a_j - s)\Gamma(s)}{\prod_{j=1}^{q}\Gamma(b_j - s)} (-z)^{-s} ds, \end{aligned} \tag{1.46}$$

where $0 < \Re s = \gamma < \min\limits_{1\leq j\leq q+1} \Re a_j$; $|\arg(-z)| < \pi$. One can find complete list of particular cases and properties of the generalized hypergeometric function in Erdélyi et al. [1], Prudnikov et al. [3], Marichev [1], Yakubovich and Luchko [2].

For our further discussions we need to note here very important particular case of function (1.45) as *hypergeometric function of Gauss*, which is defined in the unit disk as the sum of the hypergeometric series (1.45) when $p = 2, \ q = 1$, namely

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n} \frac{z^n}{n!}. \tag{1.47}$$

Its parameters a, b and c and the variable z may be complex ($c \neq 0, -1, -2, \ldots$). The series (1.47) is absolutely convergent for $|z| < 1$. It is absolutely and uniformly convergent on the circle $|z| = 1$ if $\Re(c - a - b) > 0$. For other values of z the Gauss hypergeometric function is defined as analytic continuation like (1.46). We shall explain details of such continuation below using the Slater theorem (see Marichev

[1]). Concerning the Gauss hypergeometric function (1.47) we have another method of such continuation as *the Euler integral representation*

$$ {}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt, $$

$$ 0 < \Re b < \Re c, \ |\arg(1-z)| < \pi, \tag{1.48} $$

in which the right-hand side is defined under the indicated conditions insuring the convergence of the integral. The condition $|\arg(1-z)| < \pi$ means that the function is considered in the complex plain z with the cut $(1,\infty)$, joining the singular points $z=1$ and $z=\infty$ of the Gauss hypergeometric function. It should be noted that in (1.48) we choose the principal branch $(1-tz)^{-a} = e^{-a\log(1-tz)}$, where $\log(1-tz)$ is real for $z \in [0,1]$.

One may find a most extensive list of particular cases and properties of the Gauss hypergeometric function in Erdélyi et al. [1] and Prudnikov et al. [1-3]. We put down here some of important properties of this function

$$ {}_2F_1(a,b;c;z) = {}_2F_1(b,a;c;z), \tag{1.49} $$

$$ {}_2F_1(a,b;b;z) = (1-z)^{-a}, \tag{1.50} $$

$$ {}_2F_1(a,b;c;0) = {}_2F_1(0,b;c;z) = 1, \tag{1.51} $$

$$ {}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \Re(c-a-b) > 0, \tag{1.52} $$

$$ {}_2F_1(a,b;c;z) = (1-z)^{-a}{}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right), \tag{1.53} $$

$$ {}_2F_1(a,b;c;z) = (1-z)^{c-a-b}{}_2F_1(c-a, c-b; c; z). \tag{1.54} $$

Formula (1.53) usually is called as *the Boltz formula* and relation (1.54) is called *the self-transformation formula.*

Many important special functions are defined via the Gauss hypergeometric function. Thus, we shall consider below the properties of the *associated Legendre function of the first kind* $P_\nu^\mu(z)$ represented by

$$ P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)} \left(\frac{z+1}{z-1}\right)^{\mu/2} {}_2F_1\left(-\nu, 1+\nu; 1-\mu; \frac{1-z}{2}\right), $$

$$ |\arg(z \pm 1)| < \pi, \tag{1.55} $$

$$ P_\nu^\mu(x) = \frac{1}{\Gamma(1-\mu)} \left(\frac{1+x}{1-x}\right)^{\mu/2} {}_2F_1\left(-\nu, 1+\nu; 1-\mu; \frac{1-x}{2}\right), \ |1-x| < 2 \tag{1.56} $$

and of the *associated Legendre function of the second kind* defined by

$$ Q_\nu^\mu(z) = \frac{2^{-\nu-1}}{\Gamma(\mu+3/2)} \Gamma(\mu+\nu+1) e^{i\mu\pi} \sqrt{\pi} z^{-\mu-\nu-1} (z^2-1)^{\mu/2} $$

$$ \times {}_2F_1\left((\mu+\nu+1)/2, (\mu+\nu+2)/2; \nu+3/2; z^{-2}\right), $$

$$|\arg(z \pm 1)| < \pi, \ |\arg z| < \pi, \tag{1.57}$$

$$Q^{\mu}_{\nu}(x) = \frac{1}{2} e^{-i\mu\pi}[e^{-i\mu\pi/2} Q^{\mu}_{\nu}(x + i0) + e^{i\mu\pi/2} Q^{\mu}_{\nu}(x - i0)], \quad -1 < x < 1. \tag{1.58}$$

When $\mu = 0$ by (1.55), (1.57) we define usual *Legendre functions* of the first and second kind respectively.

E. We introduced above by formula (1.46) so-called Mellin-Barnes representation or Mellin-Barnes integral for generalized hypergeometric function. Developing this approach in an attempt to give a meaning to ${}_pF_q(z)$ in the case $p > q + 1$ Meijer [1] studied the special function which is now well-known in the literature as the G-function and represented by the following Mellin-Barnes type of contour integral

$$G^{m,n}_{p,q}\left(z \middle| \begin{matrix} (\alpha_p) \\ (\beta_q) \end{matrix}\right) = G^{m,n}_{p,q}\left(z \middle| \begin{matrix} (\alpha)_{1,p} \\ (\beta)_{1,q} \end{matrix}\right) = \frac{1}{2\pi i} \int_L \Psi(s) z^{-s} ds, \tag{1.59}$$

where $z \neq 0$, $0 \le m \le q$, $0 \le n \le p$, $\alpha_j \in \mathbf{C}$, $1 \le j \le p$, $\beta_j \in \mathbf{C}$, $1 \le j \le q$,

$$\Psi(s) = \frac{\prod_{j=1}^{m} \Gamma(\beta_j + s) \prod_{j=1}^{n} \Gamma(1 - \alpha_j - s)}{\prod_{j=n+1}^{p} \Gamma(\alpha_j + s) \prod_{j=m+1}^{q} \Gamma(1 - \beta_j - s)}, \tag{1.60}$$

an empty product, if it occurs, is taken to be one, and an infinite contour L separates all left poles $s = -\beta_j - k$, $j = 1, 2, \ldots, m$, $k = 0, 1, 2, \ldots$ of the numerator from the right ones $s = 1 - \alpha_j + k$, $j = 1, 2, \ldots, n$, $k = 0, 1, 2, \ldots$ and under suitable conditions it may be one of the three types: $L_{-\infty}$, $L_{+\infty}$ or $L_{i\infty}$ (in particular, even a rectilinear line $L = (\gamma - i\infty, \gamma + i\infty)$). The description of contours and detailed list of properties and particular cases of the G-function may be found in Marichev [1], Prudnikov et al. [3], Luke [1].

We list here formulae of reflection and translation for the G-function:

$$G^{m,n}_{p,q}\left(z \middle| \begin{matrix} (\alpha_p) \\ (\beta_q) \end{matrix}\right) = G^{n,m}_{q,p}\left(\frac{1}{z} \middle| \begin{matrix} 1 - (\beta_q) \\ 1 - (\alpha_p) \end{matrix}\right), \tag{1.61}$$

$$z^{\alpha} G^{m,n}_{p,q}\left(z \middle| \begin{matrix} (\alpha_p) \\ (\beta_q) \end{matrix}\right) = G^{m,n}_{p,q}\left(z \middle| \begin{matrix} (\alpha_p) + \alpha \\ (\beta_q) + \alpha \end{matrix}\right). \tag{1.62}$$

More general function introduced by Fox [1] which is well-known in the literature as Fox's H-function or the H-function. This function is also defined by the Mellin-Barnes type of contour integral as follows

$$H^{m,n}_{p,q}\left(z \middle| \begin{matrix} (\alpha_p, a_p) \\ (\beta_q, b_q) \end{matrix}\right) = H^{m,n}_{p,q}\left(z \middle| \begin{matrix} (\alpha, a)_{1,p} \\ (\beta, b)_{1,q} \end{matrix}\right)$$

$$= \frac{1}{2\pi i} \int_L \Phi(s) z^{-s} ds, \tag{1.63}$$

where $z \neq 0$, $0 \le m \le q$, $0 \le n \le p$, $\alpha_j \in \mathbf{C}$, $a_j > 0$, $1 \le j \le p$, $\beta_j \in \mathbf{C}$, $b_j > 0$, $1 \le j \le q$,

$$\Phi(s) = \frac{\prod_{j=1}^{m} \Gamma(\beta_j + b_j s) \prod_{j=1}^{n} \Gamma(1 - \alpha_j - a_j s)}{\prod_{j=n+1}^{p} \Gamma(\alpha_j + a_j s) \prod_{j=m+1}^{q} \Gamma(1 - \beta_j - b_j s)}, \tag{1.64}$$

an empty product, if it occurs, is taken to be one, and L is a contour in the complex s-plane, which is similar to the one in relation (1.59). If all a_j, $j = 1, 2, \dots, p$, and b_j, $j = 1, 2, \dots, q$ are equal to 1, then the kernel $\Phi(s)$ (1.64) is equal to $\Psi(s)$ (1.60) and Fox's H-function (1.63) coincides with Meijer's G-function (1.59).

The H-function was studied by various mathematicians and its properties are listed, for example, in the known paper of Braaksma [1] and in the monograph by Srivastava et al. [1]. The formulae of reflection and translation for the H-function have the following form:

$$H^{m,n}_{p,q}\left(z\middle|\begin{matrix}(\alpha_p, a_p)\\(\beta_q, b_q)\end{matrix}\right) = H^{n,m}_{q,p}\left(\frac{1}{z}\middle|\begin{matrix}(1-\beta_q, b_q)\\(1-\alpha_p, a_p)\end{matrix}\right), \tag{1.65}$$

$$z^{\alpha} H^{m,n}_{p,q}\left(z\middle|\begin{matrix}(\alpha_p)\\(\beta_q)\end{matrix}\right) = H^{m,n}_{p,q}\left(z\middle|\begin{matrix}(\alpha_p+\alpha a_p, a_p)\\(\beta_q+\alpha b_q, b_q)\end{matrix}\right). \tag{1.66}$$

F. **Slater's theorem.** This theorem provides, in particular, the problem of finding the expression of the Meijer G-function (1.59) through linear combinations of generalized hypergeometric functions (1.45) with power multipliers. This expression is important for our purposes to establish the asymptotic expansions of Meijer's G-function and its particular cases by parameter. All details and proof of this theorem can be found in Slater [1] and Marichev [1].

In order to formulate Slater's theorem for our case we need to introduce some symbols. First invoking with Slater's notation (1.35) we rewrite the kernel $\Psi(s)$ by formula (1.60) as follows

$$\Psi(s) = \Gamma\left[\begin{matrix}(\beta_m)+s, 1-(\alpha_n)-s\\(\alpha_p^{n+1})+s, 1-(\beta_q^{m+1})-s\end{matrix}\right], \tag{1.67}$$

where the symbolic vectors (β_m), (α_n), (α_p^{n+1}), (β_q^{m+1}) of parameters of G-function (1.59) such that

$$\begin{aligned}(\beta_m)+s &= \beta_1+s, \dots, \ \beta_m+s,\\ 1-(\alpha_n)-s &= 1-\alpha_1-s, \dots, \ 1-\alpha_n-s,\\ (\alpha_p^{n+1})+s &= \alpha_{n+1}+s, \dots, \ \alpha_p+s,\\ 1-(\beta_q^{m+1})-s &= 1-\beta_{m+1}-s, \dots, \ 1-\beta_q-s.\end{aligned} \tag{1.68}$$

Now let us define the following key functions for the Slater theorem

$$\sum{}_m(z) = \sum_{j=1}^{m} z^{\beta_j}\Gamma\left[\begin{matrix}(\beta_m)'-\beta_j, 1-(\alpha_n)+\beta_j\\(\alpha_p^{n+1})-\beta_j, 1-(\beta_q^{m+1})+\beta_j\end{matrix}\right]$$

$$\times {}_pF_{q-1}\left[\begin{matrix}1-(\alpha_n)+\beta_j, 1-(\alpha_p^{n+1})+\beta_j;\\1-(\beta_m)'+\beta_j, 1-(\beta_q^{m+1})+\beta_j\end{matrix}(-1)^{p-n-m}z\right], \tag{1.69}$$

where $(\beta_m)' - \beta_j$ means by

$$(\beta_m)'-\beta_j = \beta_1-\beta_j, \dots, \beta_{j-1}-\beta_j, \beta_{j+1}-\beta_j, \dots, \beta_m-\beta_j, \tag{1.70}$$

and

$$\sum\nolimits_n(1/z)=\sum_{j=1}^{n} z^{\alpha_j-1}\Gamma\left[\begin{matrix}\alpha_j-(\alpha_n)',1-\alpha_j+(\beta_m)\\ \alpha_j-(\beta_q^{m+1}),1-\alpha_j+(\alpha_p^{n+1})\end{matrix}\right]$$

$$\times{}_qF_{p-1}\left[\begin{matrix}1-\alpha_j+(\beta_m),1-\alpha_j+(\beta_q^{m+1});\dfrac{(-1)^{q-n-m}}{z}\\ 1-\alpha_j+(\alpha_n)',1-\alpha_j+(\alpha_p^{n+1})\end{matrix}\right], \tag{1.71}$$

where $\alpha_j-(\alpha_n)'$ means by

$$\alpha_j-(\alpha_n)'=\alpha_j-\alpha_1,\ldots,\alpha_j-\alpha_{j-1},\alpha_j-\alpha_{j+1},\ldots,\alpha_j-\alpha_n. \tag{1.72}$$

Theorem 1.6. *Let the following groups of conditions for the kernel $\Psi(s)$ defined by (1.60) hold*

$$-\Re\beta_j<\Re s<1-\Re\alpha_k,\ j=1,2,\ldots,m,\quad k=1,2,\ldots,n, \tag{1.73}$$

$$2(m+n)\geq p+q, \tag{1.74}$$

$$(q-p)\Re s<-\Re\varrho,\ if\ 2(m+n)=p+q, \tag{1.75}$$

where

$$\varrho=\sum_{j=1}^{q}\beta_j-\sum_{k=1}^{p}\alpha_k+\frac{p-q}{2}, \tag{1.76}$$

$$\Re\varrho<0,\ if\ p=q=m+n. \tag{1.77}$$

Then for real argument $x>0$ Meijer's G-function (1.59) equals to one of the following expressions through functions (1.69), (1.71)

$$G^{m,n}_{p,q}\left(x\middle|\begin{matrix}(\alpha_p)\\(\beta_q)\end{matrix}\right)=\sum\nolimits_m(x),\quad x>0\ if\ q>p, \tag{1.78}$$

$$G^{m,n}_{p,q}\left(x\middle|\begin{matrix}(\alpha_p)\\(\beta_q)\end{matrix}\right)=\sum\nolimits_m(x),\quad 0<x<1\ if\ q=p, \tag{1.79}$$

$$G^{m,n}_{p,q}\left(x\middle|\begin{matrix}(\alpha_p)\\(\beta_q)\end{matrix}\right)=\sum\nolimits_n(1/x),\quad x>1\ if\ q=p, \tag{1.80}$$

$$G^{m,n}_{p,q}\left(x\middle|\begin{matrix}(\alpha_p)\\(\beta_q)\end{matrix}\right)=\sum\nolimits_n(1/x),\quad x>0\ if\ q<p. \tag{1.81}$$

Moreover, when $q=p$ and $m+n>p$ then functions (1.69), (1.71) are analytic continuations of one another as the functions of complex variable z and the equality at the point $x=1$ is true

$$G^{m,n}_{p,q}\left(1\middle|\begin{matrix}(\alpha_p)\\(\beta_q)\end{matrix}\right)=\sum\nolimits_m(1)=\sum\nolimits_n(1) \tag{1.82}$$

under conditions

$$p=q,\ \Re\varrho+p-m-n+1<0,\quad m+n\geq p. \tag{1.83}$$

Theorem 1.6 is a particular case of the Slater theorem concerning representations of the Meijer G-function (1.59), which are useful for our further purposes. More general Mellin-Barnes representations and the proof of the Slater theorem the reader can find in books of Slater [1] and Marichev [1]. Inversely, the Slater theorem mentioned above gives other expressions of hypergeometric functions through the Mellin-Barnes integrals (1.60) and allows us to find their analytic continuations. We shall list below some special functions which we need in terms of Meijer's G-functions.

Let us return to the Gauss hypergeometric function (1.47) and observe that by the Slater theorem one can introduce an equivalent definition of the Gauss function in terms of Mellin-Barnes integrals, namely

$$\frac{\Gamma(a)\Gamma(b)}{\Gamma(c)}{}_2F_1(a,b;c;z)=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(s)\Gamma(a-s)\Gamma(b-s)}{\Gamma(c-s)}(-z)^{-s}ds$$

$$=G^{1,2}_{2,2}\left(-z\left|\begin{matrix}1-a, & 1-b\\ 0, & 1-c\end{matrix}\right.\right),\ |\arg(-z)|<\pi,\quad |z|<1. \tag{1.84}$$

This case related to formula (1.79), i.e. usual Gauss's function (1.47) is the infinite sum of residues (see (1.28)) of the integrand (1.84) at the left poles $s=-n$, $n=0,1,\ldots$ of gamma-function $\Gamma(s)$. But due to formula (1.80) we can regard the integral (1.84) for $|z|>1$ also as a sum of "right" residues. Then, if there were no multiple poles, that is, $a-b$ is not an integer, the integral (1.84) can be evaluated by formula (1.71) which gives the value of the Gauss hypergeometric function as

$${}_2F_1(a,b;c;z)=\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)^{-a}{}_2F_1(a,1-c+a;1-b+a;1/z)$$

$$+\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}(-z)^{-b}{}_2F_1(b,1-c+b;1-a+b;1/z), \tag{1.85}$$

where $|z|>1$, $a-b\neq k$, $k=0,\pm1,\ \pm2,\ldots,|\arg(-z)|<\pi$. Formula (1.85) shows that for large $|z|$, when $a-b$ is not an integer, Gauss's function has the estimate

$${}_2F_1(a,b;c;z)=C_1z^{-a}+C_2z^{-b}+O(z^{-a-1})+O(z^{-b-1}), \tag{1.86}$$

where $C_1,\ C_2$ are constants.

G. We define *the Bessel functions* $J_\nu(z),\ Y_\nu(z),\ I_\nu(z),\ K_\nu(z)$ based on the hypergeometric series

$${}_0F_1\,(c;\ z)=\sum_{n=0}^{\infty}\frac{z^n}{(c)_n n!}\ |z|<\infty, \tag{1.87}$$

by the following formulae

$$J_\nu(z)=\frac{1}{\Gamma(\nu+1)}\left(\frac{z}{2}\right)^{\nu}{}_0F_1\left(\nu+1;\ -\frac{z^2}{4}\right)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n+\nu}}{\Gamma(\nu+n+1)n!} \tag{1.88}$$

(*the Bessel function of the first kind*),

$$Y_\nu(z) = \frac{1}{\sin \pi\nu} \left[J_\nu(z) \cos \pi\nu - J_{-\nu}(z)\right],$$

$$Y_n(z) = \lim_{\nu \to n} Y_\nu(z), \; n = 0, \pm 1, \pm 2, \ldots \tag{1.89}$$

(*the Bessel function of the second kind or the Neumann function*),

$$I_\nu(z) = \sum_{n=0}^{\infty} \frac{(z/2)^{2n+\nu}}{\Gamma(\nu+n+1)n!} = e^{-\pi i\nu/2} J_\nu(iz) \tag{1.90}$$

(*the modified Bessel function*),

$$K_\nu(z) = \frac{\pi}{2\sin(\pi\nu)} \left[I_{-\nu}(z) - I_\nu(z)\right], \; \nu \neq 0, \pm 1, \pm 2, \ldots,$$

$$K_n(z) = \lim_{\nu \to n} K_\nu(z), \; n = 0, \pm 1, \pm 2, \ldots \tag{1.91}$$

(*the Macdonald function*). It is clear that $K_{-\nu}(z) = K_\nu(z)$.

Bessel functions $J_\nu(x)$, $I_\nu(x)$, $K_\nu(x)$ have the following asymptotic behavior (see Erdélyi et al. [1]):

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}} \cos(x - \pi(1+2\nu)/4) + O(x^{-3/2}), \; x \to +\infty, \tag{1.92}$$

$$J_\nu(x) = O(x^{\Re(\nu)}), \; x \to 0+, \tag{1.93}$$

$$I_\nu(x) = O(e^x/\sqrt{2\pi x}), \; x \to +\infty, \tag{1.94}$$

$$I_\nu(x) = O(x^{\Re(\nu)}), \; \nu \neq 0, \; x \to 0+, \tag{1.95}$$

$$K_\nu(x) = O\left(e^{-x}\sqrt{\frac{\pi}{2x}}\right), \; x \to +\infty, \tag{1.96}$$

$$K_\nu(x) = O(x^{-|\Re(\nu)|}), \; \nu \neq 0, \; K_0(x) = O(\log x), \; x \to 0+. \tag{1.97}$$

Here we also note some integral representations connected with the Macdonald function of pure imaginary index $K_{i\tau}(x)$, $x > 0$, $\tau \in \mathbf{R}$. Sometimes it is sufficient to consider $\tau \geq 0$ due to the evenness of the Macdonald function by its index. We need to give also some integrals, which involve the Legendre functions mentioned above to use theirs in further considerations. Thus we have the following formula (see Erdélyi et al. [1])

$$K_{i\tau}(x) = \int_0^\infty e^{-x\cosh u} \cos \tau u du$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} e^{-x\cosh u} e^{i\tau u} du, \; x > 0. \tag{1.98}$$

By the analytic property of the integrand in (1.98) and by its asymptotic behavior at the contour we can shift it along the horizontal open infinite strip $(i\delta - \infty,\ i\delta + \infty)$ with $\delta \in [0,\ \pi/2)$ as

$$K_{i\tau}(x) = \frac{1}{2}\int_{i\delta-\infty}^{i\delta+\infty} e^{-x\cosh\beta} e^{i\tau\beta} d\beta,\ x > 0. \tag{1.99}$$

From representation (1.99) we easily obtain useful uniform estimate of the Macdonald function $K_{i\tau}(x)$ by its index $\tau > 0$ and argument $x > 0$, namely

$$|K_{i\tau}(x)| \le \frac{1}{2} e^{-\delta\tau} \int_{-\infty}^{\infty} e^{-x\cos\delta\cosh u} du$$

$$= e^{-\delta\tau} K_0(x\cos\delta), \quad \delta \in [0,\ \pi/2). \tag{1.100}$$

Let us give other integral formulae from Prudnikov et al. [2], which involve the Macdonald function $K_{i\tau}(x)$ and the Gauss, Legendre, Bessel functions considered above. Indeed, by formula 2.16.21.1 from Prudnikov et al. [2] we have

$$\int_0^\infty y^{\alpha-1} J_\mu(xy) K_{i\tau}(y) dy = 2^{\alpha-2} x^\mu \frac{\Gamma((\alpha+\mu+i\tau)/2)\Gamma((\alpha+\mu-i\tau)/2)}{\Gamma(\mu+1)}$$

$$\times {}_2F_1\left(\frac{\alpha+\mu+i\tau}{2},\ \frac{\alpha+\mu-i\tau}{2};\ \mu+1; -x^2\right),\quad \Re(\alpha+\mu) > 0. \tag{1.101}$$

This integral contains as particular cases formulae for the Legendre functions (see (1.55), (1.56)), in which we shall need later. For the Legendre function of the first kind it has own integral representation by formula 2.16.6.3 from Prudnikov et al. [2], namely

$$\int_0^\infty y^{\alpha-1} e^{-xy} K_{i\tau}(y) dy = \sqrt{\frac{\pi}{2}}\Gamma(\alpha+i\tau)\Gamma(\alpha-i\tau)$$

$$\times (1-x^2)^{(1-2\alpha)/4} P^{1/2-\alpha}_{-1/2+i\tau}(x),\ x > 0,\ \Re\alpha > 0. \tag{1.102}$$

We note the Macdonald formula from Erdélyi et al. [1] (see also Prudnikov et al. [2]) as

$$K_\nu(x)K_\nu(y) = \frac{1}{2}\int_0^\infty \exp\left(-\frac{1}{2}\left(\frac{xy}{u} + \frac{xu}{y} + \frac{uy}{x}\right)\right) K_\nu(u)\frac{du}{u},\ \nu \in \mathbf{C}. \tag{1.103}$$

The following integral 2.2.4.5 from Prudnikov et al. [2] by arguments of the Euler gamma-functions is useful in further considerations too

$$\int_0^\infty \Gamma\left(\frac{s}{2} + \frac{it}{2}\right)\Gamma\left(\frac{s}{2} - \frac{it}{2}\right)\cos(\tau t) dt = \frac{\pi}{2^{s-1}}\Gamma(s)\cosh^{-s}\tau,\ \Re s > 0. \tag{1.104}$$

We include here two conditionally convergent integrals from Erdélyi et al. [1], which are worth mentioning

$$\cosh\left(\frac{\pi\tau}{2}\right) K_{i\tau}(x) = \int_0^\infty \cos(x\sinh u)\cos(\tau u) du,\ x > 0, \tag{1.105}$$

$$\sinh\left(\frac{\pi\tau}{2}\right) K_{i\tau}(x) = \int_0^\infty \sin(x \sinh u)\sin(\tau u)du, \ x > 0. \tag{1.106}$$

Now to introduce some special functions of hypergeometric type we use the Mellin-Barnes integral (1.59) for the Meijer G-function of a real argument $x > 0$ and list its important particular cases based on the table from Prudnikov et al. [3] as

the Bessel functions

$$G^{1,0}_{0,2}\left(x\middle|_{\nu/2,\ -\nu/2}\right) = J_\nu(2\sqrt{x}), \tag{1.107}$$

$$2\frac{\cos(\pi\nu)}{\sqrt{\pi}}G^{2,1}_{2,4}\left(x\middle|{}^{1/2,\ 0}_{\nu,\ -\nu,\ 0,\ 0}\right) = J^2_{-\nu}(\sqrt{x}) + J^2_\nu(\sqrt{x}), \tag{1.108}$$

$$2\frac{\sin(\pi\nu)}{\sqrt{\pi}}G^{2,0}_{1,3}\left(x\middle|{}^{1/2}_{\nu,\ -\nu,\ 0}\right) = J^2_{-\nu}(\sqrt{x}) - J^2_\nu(\sqrt{x}), \tag{1.109}$$

$$\begin{aligned}&\frac{\sin(2\pi\nu)}{\pi^{5/2}}G^{3,1}_{1,3}\left(x\middle|{}^{1/2}_{0,\nu,\ -\nu}\right)\\ &= J_\nu(\sqrt{x})Y_{-\nu}(\sqrt{x}) - J_{-\nu}(\sqrt{x})Y_\nu(\sqrt{x}),\end{aligned} \tag{1.110}$$

$$\begin{aligned}&\frac{2\cos(\pi\nu)}{\pi^{5/2}}G^{3,1}_{1,3}\left(x\middle|{}^{1/2}_{0,\nu,\ -\nu}\right)\\ &= J^2_\nu(\sqrt{x}) + Y^2_\nu(\sqrt{x}),\end{aligned} \tag{1.111}$$

$$\begin{aligned}&\frac{\sin(2\pi\nu)}{\pi^{3/2}}G^{2,1}_{1,3}\left(x\middle|{}^{1/2}_{\nu,\ -\nu,\ 0}\right)\\ &= I^2_{-\nu}(\sqrt{x}) - I^2_\nu(\sqrt{x}),\end{aligned} \tag{1.112}$$

$$\frac{1}{2}G^{2,0}_{0,2}\left(x\middle|_{\nu/2,\ -\nu/2}\right) = K_\nu(2\sqrt{x}), \tag{1.113}$$

$$\sqrt{\pi}G^{2,0}_{1,2}\left(x\middle|{}^{1/2}_{\nu,\ -\nu}\right) = e^{-x/2}K_\nu\left(\frac{x}{2}\right), \tag{1.114}$$

$$\frac{\cos(\pi\nu)}{\sqrt{\pi}}G^{2,1}_{1,2}\left(x\middle|{}^{1/2}_{\nu,\ -\nu}\right) = e^{x/2}K_\nu\left(\frac{x}{2}\right), \tag{1.115}$$

$$\begin{aligned}&\frac{\cos(\pi\nu)}{\sqrt{\pi}}G^{2,1}_{1,3}\left(x\middle|{}^{1/2}_{\nu,\ -\nu,\ 0}\right)\\ &= [I_{-\nu}(\sqrt{x}) + I_\nu(\sqrt{x})]K_\nu(\sqrt{x}),\end{aligned} \tag{1.116}$$

$$\frac{\sqrt{\pi}}{2}G^{3,0}_{1,3}\left(x\middle|{}^{1/2}_{0,\ \nu,\ -\nu}\right) = K^2_\nu(\sqrt{x}); \tag{1.117}$$

the Lommel functions

$$2^{\mu-1}\Gamma\left(\frac{\mu-\nu+1}{2}\right)\Gamma\left(\frac{\mu+\nu+1}{2}\right)G^{1,1}_{1,3}\left(x\middle|{}^{(\mu+1)/2}_{(\mu+1)/2,\ \nu/2,\ -\nu/2}\right)$$

$$= s_{\mu,\nu}(2\sqrt{x}), \tag{1.118}$$

$$\frac{2^{\mu-1}}{\Gamma((1-\mu-\nu)/2)\Gamma((1-\mu+\nu)/2)} G^{3,1}_{1,3}\left(x\middle|\begin{matrix}(\mu+1)/2\\(\mu+1)/2,\ \nu/2,\ -\nu/2\end{matrix}\right)$$
$$= S_{\mu,\nu}(2\sqrt{x}); \tag{1.119}$$

the Thompson's functions

$$\frac{1}{8\sqrt{\pi}} G^{4,0}_{0,4}\left(x\middle|\begin{matrix}\\0,\ 1/2,\ \nu/2,\ -\nu/2\end{matrix}\right)$$
$$= \mathrm{ker}^2_\nu(2(4x)^{1/4}) + \mathrm{kei}^2_\nu(2(4x)^{1/4}); \tag{1.120}$$

the Legendre functions

$$G^{2,0}_{2,2}\left(x\middle|\begin{matrix}1,\ 1-\mu\\-\nu,\ 1+\nu\end{matrix}\right) = H(1-x)(1-x)^{-\mu/2} P^\mu_\nu\left(\frac{2}{x}-1\right),\ \Re\mu < 1, \tag{1.121}$$

where $H(x)$ is the Heaviside function;

$$-\frac{\sin(\pi\nu)}{\pi} G^{1,2}_{2,2}\left(x\middle|\begin{matrix}1+\nu-\mu/2,\ -\nu-\mu/2\\-\mu/2,\ \mu/2\end{matrix}\right) = (1+x)^{-\mu/2} P^\mu_\nu(2x+1), \tag{1.122}$$

$$\frac{2^{\mu+1}\pi}{\Gamma(1-(\mu-\nu)/2)\Gamma((1-\mu-\nu)/2)} G^{2,0}_{2,2}\left(x\middle|\begin{matrix}(1+\nu)/2,\ -\nu/2\\\mu/2,\ -\mu/2\end{matrix}\right)$$
$$= H(1-x)(1-x)^{-1/2}\left[P^\mu_\nu(\sqrt{1-x}) + P^\mu_\nu(-\sqrt{1-x})\right], \tag{1.123}$$

$$\frac{1}{\sqrt{\pi}\Gamma(1-\mu+\nu)\Gamma(-\mu-\nu)} G^{1,3}_{3,3}\left(x\middle|\begin{matrix}-\nu,\ 1+\nu,\ 1/2\\-\mu,\ 0,\ \mu\end{matrix}\right)$$
$$= \left[P^\mu_\nu(\sqrt{1+x})\right]^2, \tag{1.124}$$

$$-\frac{\sin(\pi\nu)}{\pi^{3/2}} G^{3,1}_{3,3}\left(x\middle|\begin{matrix}-\nu,\ 1+\nu,\ 1/2\\0,\ -\mu,\ \mu\end{matrix}\right)$$
$$= P^{-\mu}_\nu(\sqrt{1+x}) P^\mu_\nu(\sqrt{1+x}), \tag{1.125}$$

$$\frac{1}{\sqrt{\pi}\Gamma(1-\mu+\nu)\Gamma(1-\mu-\nu)} G^{1,3}_{3,3}\left(x\middle|\begin{matrix}\nu,\ -\nu,\ 1/2\\-\mu,\ 0,\ \mu\end{matrix}\right)$$
$$= \frac{P^\mu_\nu(\sqrt{1+x}) P^\mu_{-\nu}(\sqrt{1+x})}{\sqrt{1+x}}, \tag{1.126}$$

$$2^{\mu-1} e^{i\mu\pi} \frac{\Gamma(1+(\mu+\nu)/2)}{\Gamma((1-\mu+\nu)/2)} G^{2,1}_{2,2}\left(x\middle|\begin{matrix}(1-\nu)/2,\ 1+\nu/2\\\mu/2,\ -\mu/2\end{matrix}\right)$$
$$= Q^\mu_\nu(\sqrt{1+x}), \tag{1.127}$$

$$\frac{e^{i\mu\pi}}{\sqrt{\pi}} \cos(\mu\pi)\Gamma(\mu+\nu+1) G^{2,1}_{2,2}\left(x\middle|\begin{matrix}1/2,\ 1+\nu\\\mu,\ -\mu\end{matrix}\right)$$

$$= (1+x)^{\nu/2} Q_\nu^\mu \left(\frac{2+x}{2\sqrt{1+x}} \right), \tag{1.128}$$

$$\frac{\sqrt{\pi}}{2} e^{2i\mu\pi} \frac{\Gamma(1+\mu+\nu)}{\Gamma(1-\mu+\nu)} G^{3,1}_{3,3} \left(x \middle| \begin{matrix} -\nu, \ 1/2, \ 1+\nu \\ 0, \ \mu, \ -\mu \end{matrix} \right)$$

$$= \left[Q_\nu^\mu(\sqrt{1+x}) \right]^2; \tag{1.129}$$

the Whittaker functions

$$2\sqrt{\pi} \frac{\Gamma^2(2\sigma+1)}{\Gamma(1/2-\rho+\sigma)\Gamma(1/2+\rho+\sigma)} G^{2,1}_{4,2} \left(x \middle| \begin{matrix} 1/2-\sigma, \ 1/2+\sigma, \ 0, 1/2 \\ \rho, \ -\rho \end{matrix} \right)$$

$$= M_{\rho,\,\sigma}\left(\frac{2i}{\sqrt{x}}\right) M_{\rho,\,\sigma}\left(\frac{-2i}{\sqrt{x}}\right), \tag{1.130}$$

$$G^{2,0}_{1,2} \left(x \middle| \begin{matrix} 1-\rho \\ 1/2+\sigma, \ 1/2-\sigma \end{matrix} \right) = e^{-x/2} W_{\rho,\,\sigma}(x), \tag{1.131}$$

$$\frac{1}{\sqrt{\pi}} G^{4,0}_{4,2} \left(x \middle| \begin{matrix} 1+\rho, \ 1-\rho \\ 1/2, \ 1, \ 1/2+\sigma, \ 1/2-\sigma \end{matrix} \right)$$

$$= W_{-\rho,\,\sigma}\left(2\sqrt{x}\right) W_{\rho,\,\sigma}\left(2\sqrt{x}\right); \tag{1.132}$$

the Kummer function

$$\frac{\Gamma(b)}{\Gamma(a)} G^{1,1}_{1,2} \left(x \middle| \begin{matrix} 1-a \\ 0, \ 1-b \end{matrix} \right) = {}_1F_1(a;\ b;\ -x), \ b \neq 0, \ -1, \ -2, \ldots; \tag{1.133}$$

the Tricomi function

$$\frac{1}{\Gamma(a)\Gamma(a-b+1)} G^{2,1}_{1,2} \left(x \middle| \begin{matrix} 1-a \\ 0, \ 1-b \end{matrix} \right) = \Psi(a;\ b;\ x); \tag{1.134}$$

the Gauss hypergeometric function

$$\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} G^{1,2}_{2,2} \left(x \middle| \begin{matrix} 1-a, \ 1-b \\ 0, \ 1-c \end{matrix} \right) = {}_2F_1(a,\ b;\ c;\ -x), \ c \neq 0, \ -1, \ -2, \ldots, \tag{1.135}$$

$$\frac{2^{2a+2b-1}}{\sqrt{\pi}} \frac{\Gamma^2(a+b+1/2)}{\Gamma(2a)\Gamma(2b)} G^{3,1}_{3,3} \left(x \middle| \begin{matrix} 1, \ 1/2+a+b, \ 2a+2b \\ 2a, \ 2b, \ a+b \end{matrix} \right)$$

$$= {}_2F_1^2\left(a,\ b;\ a+b+1/2;\ -\frac{1}{x}\right), \tag{1.136}$$

$$\frac{2^{2a+2b-1}}{\sqrt{\pi}} \frac{\Gamma^2((a+b+1)/2)}{\Gamma(a)\Gamma(b)} G^{3,1}_{3,3} \left(x \middle| \begin{matrix} 1, \ a+b, \ (a+b+1)/2 \\ a, \ b, \ (a+b)/2 \end{matrix} \right)$$

$$= {}_2F_1^2\left(a,\ b;\ \frac{a+b+1}{2};\ -\frac{\sqrt{x}-\sqrt{1+x}}{2\sqrt{x}}\right), \tag{1.137}$$

$$\frac{(1-2a-2b)\cos(a-b)\pi}{2\sqrt{\pi}\cos(a+b)\pi} G^{3,1}_{3,3} \left(x \middle| \begin{matrix} 1, \ \frac{1}{2}+a+b, \ \frac{3}{2}-a-b \\ \frac{1}{2}, \ \frac{1}{2}+a-b, \ \frac{1}{2}+b-a \end{matrix} \right)$$

$$= {}_2F_1\left(a,\ b;\ a+b+\frac{1}{2};\ -\frac{1}{x}\right){}_2F_1\left(\frac{1}{2}-a,\ \frac{1}{2}-b;\ \frac{3}{2}-a-b;\ -\frac{1}{x}\right); \tag{1.138}$$

the Clausen function

$$\frac{\Gamma(b_1)\Gamma(b_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} G^{1,3}_{3,3}\left(x\left|\begin{matrix}1-a_1,\ 1-a_2,\ 1-a_3\\ 0,\ 1-b_1,\ 1-b_2\end{matrix}\right.\right)$$

$$= {}_3F_2(a_1,\ a_2,\ a_3;\ b_1,\ b_2;\ -x); \tag{1.139}$$

the Appel F_3-function

$$\Gamma(c)G^{0,3}_{3,3}\left(x\left|\begin{matrix}a_1+b_1,\ c-a,\ c-b\\ a_1,\ b_1,\ c-a-b\end{matrix}\right.\right)$$

$$= H(x-1)(x-1)^{c-1}F_3\left(a,\ a_1,\ b,\ b_1,\ c;\ 1-x,\ 1-\frac{1}{x}\right). \tag{1.140}$$

The last part of this section deals with the asymptotic expansions of some special functions mentioned above by their index or parameter. Conventionally, the respective questions are more popular for the asymptotic expansions of special functions by argument and already are investigated in detail (see Erdélyi et al. [1], Olver [1]). Nevertheless, we can find several formulae of asymptotic expansions, for instance of the Bessel functions by index in Erdélyi et al. [1] and Lebedev [9] and these questions are worth mentioning for further investigation of the index integrals. Here we give rigorous proofs of such formulae extensively used the Stirling formula (1.32) of the asymptotic expansion of the gamma-function. Thus we shall call such formulae as the Stirling type formulae.

First we need to estimate the remainder term of the Stirling formula (1.32) starting from the formula of Binet (1.34). Indeed, taking $z = \alpha + i\tau$, where $\alpha > 0$ denote by

$$\varphi(\alpha+i\tau) = \int_0^\infty \left[\frac{1}{2}+\frac{1}{t}-\frac{1}{1-e^{-t}}\right]\frac{e^{-(\alpha+i\tau)t}}{t}dt$$

$$= \int_0^\infty \psi(t)t^{-\gamma}e^{-i\tau t}dt,\ 0<\gamma<1. \tag{1.141}$$

Here we mean that

$$\psi(t) = \left[\frac{1}{2}+\frac{1}{t}-\frac{1}{1-e^{-t}}\right]t^{-\beta}e^{-\alpha t},\ \beta = 1-\gamma. \tag{1.142}$$

It is evident, that $\psi(t)$ is of bounded variation in $(0,\infty)$. Moreover, we have the following asymptotic relations

$$\psi(t) = O\left(t^\gamma\right),\ t\to 0+, \tag{1.143}$$

$$\psi(t) = O\left(t^{-\beta}e^{-\alpha t}\right),\ t\to +\infty. \tag{1.144}$$

Now one can use the result from Titchmarsh [1, Theorem 126] to estimate the function $\varphi(\alpha + i\tau)$, when $\tau \to +\infty$ for all $\alpha > 0$ uniformly, taking into account asymptotic relations (1.143)-(1.144). Thus we obtain

$$\varphi(\alpha + i\tau) = \int_0^\infty \psi(t)t^{-\gamma}\cos(\tau t)dt$$

$$+i\int_0^\infty \psi(t)t^{-\gamma}\sin(\tau t)dt = O(1/\tau), \ \tau \to +\infty. \tag{1.145}$$

So one can rewrite Stirling's formula (1.32) for our further purposes using the Binet formula (1.34) and the derived asymptotic (1.145) as follows

$$\Gamma(\alpha+i\tau) = \sqrt{2\pi}\exp\left[\left(\alpha - \frac{1}{2} + i\tau\right)\left(\log\sqrt{\tau^2+\alpha^2} + i\tan^{-1}(\tau/\alpha)\right) - \alpha - i\tau - O(1/\tau)\right]$$

$$= \sqrt{2\pi}\tau^{\alpha-1/2}e^{-\frac{\pi\tau}{2}}\exp\left[i\left(\frac{(\alpha-1/2)\pi}{2} + \tau\log\tau - \tau\right) - O(1/\tau)\right], \ \tau \to +\infty. \tag{1.146}$$

Note here that in the case $\alpha \le 0$ it is not difficult to write asymptotic expansion of the respective gamma-function using formula (1.146) and expressing it by means of supplement formula (1.29). Moreover, without loss of generality we shall mean that free parameters of gamma-functions $\Gamma(\alpha + i\tau)$ are real numbers. Let us give the asymptotic expansion of the Macdonald function (1.98) for $|\tau| \to \infty$. As it is evident from integral (1.98) this function is continuous by index $\tau \in \mathbf{R}$ due to its absolute and uniform convergence which gives rough (comparing with (1.100)) but useful inequality

$$|K_{i\tau}(x)| \le K_0(x), \ \tau \in \mathbf{R}, \ x > 0. \tag{1.147}$$

Due to the evenness of the Macdonald function by its index we can consider index $\tau \to +\infty$.

Theorem 1.7. *For the Macdonald function $K_{i\tau}(x)$ by an arbitrary variable $x \in (0, X]$, $X > 0$ the next asymptotic expansion by its index is true*

$$K_{i\tau}(x) = \sqrt{\frac{2\pi}{\tau}}e^{-\pi\tau/2}\sin\left(\tau\log\frac{2\tau}{x} - \tau + \frac{\pi}{4} + \frac{x^2}{4\tau}\right)$$

$$\times\left[1 + O(1/\tau)\right], \ \tau \to +\infty. \tag{1.148}$$

Proof. By definition of the Macdonald function (1.91) use series (1.90) for the Bessel function $I_{i\tau}(x)$, which is absolutely and uniformly convergent by $\tau > 0$ and $0 < x \le X$ as it is easily seen from the estimate (accounting inequality (1.44))

$$| I_{i\tau}(x) | \le \sqrt{\pi}\left|\frac{1}{\Gamma(1/2+i\tau)}\right|\sum_{k=0}^\infty \frac{(X/2)^{2k}}{k!\Gamma(k+1)}. \tag{1.149}$$

Therefore, setting $\alpha = k+1$ at formula (1.46) and applying it to the Bessel function $I_{i\tau}(x)$ we have the following asymptotic expansion

$$I_{i\tau}(x) = \lim_{M\to\infty} \sum_{k=0}^{M} \frac{(x/2)^{2k+i\tau}}{k!\Gamma(k+i\tau+1)}$$

$$= \frac{e^{\frac{\pi\tau}{2}}}{\sqrt{2\pi\tau}} \exp\left[-i\left(\tau\log\frac{2\tau}{x} - \tau + \frac{\pi}{4}\right)\right](1+O(1/\tau)) \lim_{M\to\infty}\sum_{k=0}^{M} \frac{\left(e^{-i\pi/2}\frac{x^2}{4\tau}\right)^k}{k!}. \quad (1.150)$$

We note that it is possible to carry out of the limit asymptotic terms in (1.150), because for sufficiently large τ the remainder in Stirling's formula becomes $O(1/\tau)$ uniformly for $k \in [0,\ M]$. Hence passing to the limit at the series due its absolute and uniform convergence we obtain the following expansion

$$I_{i\tau}(x) = \frac{e^{\frac{\pi\tau}{2}}}{\sqrt{2\pi\tau}} \exp\left[-i\left(\tau\log\frac{2\tau}{x} - \tau + \frac{\pi}{4} + \frac{x^2}{4\tau}\right)\right](1+O(1/\tau)), \ \tau \to +\infty. \quad (1.151)$$

For the Bessel function $I_{-i\tau}(x)$ we have similar (1.151) expansion as

$$I_{-i\tau}(x) = \frac{e^{\frac{\pi\tau}{2}}}{\sqrt{2\pi\tau}} \exp\left[i\left(\tau\log\frac{2\tau}{x} - \tau + \frac{\pi}{4} + \frac{x^2}{4\tau}\right)\right](1+O(1/\tau)) \ \tau \to +\infty, \quad (1.152)$$

invoking with the asymptotic value of the respective gamma-function, namely

$$\Gamma(k+1-i\tau) = \sqrt{2\pi}\tau^{k+1/2}e^{-\frac{\pi\tau}{2}}$$

$$\times \exp\left[-i\left(\frac{(k+1/2)\pi}{2} + \tau\log\tau - \tau\right) - O(1/\tau)\right], \ \tau \to +\infty. \quad (1.153)$$

Substituting the obtained results into formula (1.91) with $\nu = i\tau$ and making use the supplement formula (1.29) for the gamma-function to express the multiplier $\pi(\sin(i\pi\tau))^{-1}$ as

$$\frac{\pi}{\sin(i\tau\pi)} = \Gamma(i\tau)\Gamma(1-i\tau) = 2\pi e^{-\pi\tau - i\pi/2}(1+O(1/\tau)), \ \tau \to +\infty \quad (1.154)$$

we lead to formula (1.148). Theorem 1.7 is proved. •

Now we turn to the asymptotic expansion by index $\nu = -1/2 + i\tau$ of the associated Legendre function of the first kind defined by formulae (1.55)-(1.56). Its expansion depends upon the range of variable x as we see from formulae (1.47) and (1.85) for the Gauss hypergeometric function and its analytic continuation.

Theorem 1.8. *For the associated Legendre function of the first kind* $P^{\mu}_{-1/2+i\tau}(x)$ *by arbitrary variable* $x > 1$ *the next asymptotic expansions by its index* $\tau \to +\infty$ *take place*

$$P^{\mu}_{-1/2+i\tau}(x) = \left(\frac{x+1}{2}\right)^{\mu/2} \tau^{\mu} J_{-\mu}(\tau\sqrt{2(x-1)})(1+O(1/\tau)), \ 1 < x < 3; \quad (1.155)$$

$$P^{\mu}_{-1/2+i\tau}(x) = \sqrt{\frac{2}{\pi}} \left(\frac{x+1}{x-1}\right)^{\mu/2} \frac{\tau^{\mu-1/2}}{\sqrt{x-1}}$$

$$\times \cos\left(\tau \log(2(x-1)) + (\mu - 1/2)\frac{\pi}{2} + \frac{\tau}{x-1}\right) (1 + O(1/\tau)),\ x > 3; \tag{1.156}$$

$$P^{\mu}_{-1/2+i\tau}(3) = \tau^{\mu} 2^{i\tau - 1/2} J_{-\mu}(\tau\sqrt{2}) \left[1 + O(1/\tau)\right],\ \mu < 0, \tag{1.157}$$

where $J_\nu(z)$ *is the Bessel function* (1.88).

Proof. Let us prove formula (1.155). For our purpose we consider the values $x > 1$. So representation (1.56) involves the case $1 < x < 3$. Putting $\nu = -1/2 + i\tau$ at (1.56) use series (1.47) for the Gauss hypergeometric function. By definition (1.36) of the Pochhammer symbol with inequality (1.26) from (1.56) we have the uniform estimate for $\mid 1 - x \mid < x_0 < 2$ as

$$\left| {}_2F_1\left(1/2 - i\tau, 1/2 + i\tau; 1 - \mu; \frac{1-x}{2}\right) \right|$$

$$\leq \frac{1}{\mid \Gamma(1/2 - i\tau)\Gamma(1/2 + i\tau) \mid} \sum_{n=0}^{\infty} \frac{\Gamma^2(1/2 + n)}{\mid (1-\mu)_n \mid n!} \left(\frac{x_0}{2}\right)^n. \tag{1.158}$$

Formula (1.146) gives asymptotic expansions of the Pochhammer symbols and we have

$$(1/2 - i\tau)_n (1/2 + i\tau)_n = \tau^{2n}(1 + O(1/\tau)),\ \tau \to +\infty. \tag{1.159}$$

Thus we obtain the following asymptotic equality

$$P^{\mu}_{-1/2+i\tau}(x) = \frac{1}{\Gamma(1-\mu)} \left(\frac{x+1}{x-1}\right)^{\mu/2}$$

$$\times \lim_{M \to \infty} \sum_{n=0}^{M} \frac{\left(\tau^2 \frac{1-x}{2}\right)^n}{(1-\mu)_n n!} \left[1 + O(1/\tau)\right],\ \tau \to +\infty. \tag{1.160}$$

Since the remainder $O(1/\tau)$ is independent from k for $\tau \to +\infty$ one can pass to the limit and invoking with relation for the Bessel function (1.88) we arrive to equality

$$\lim_{M \to \infty} \sum_{n=0}^{M} \frac{\left(\tau^2 \frac{1-x}{2}\right)^n}{(1-\mu)_n n!} = {}_0F_1\left(1 - \mu; \tau^2 \frac{1-x}{2}\right)$$

$$= \Gamma(1-\mu)\tau^{\mu}((x-1)/2)^{\mu/2} J_{-\mu}(\tau\sqrt{2(x-1)}). \tag{1.161}$$

By substituting it into (1.160) we immediately establish (1.155).

The asymptotic expansion for $\tau \to +\infty$ at the point $x = 3$ given by formula (1.157) follows from Boltz's formula (1.53). Indeed, for $\mu < 0$ series (1.56) keeps as an absolutely convergent at the point $x = 3$ and we have from formula (1.53) that

$${}_2F_1\left(1/2 - i\tau, 1/2 + i\tau; 1 - \mu; -1\right) = 2^{i\tau - 1/2} {}_2F_1\left(1/2 - i\tau, 1/2 - \mu - i\tau; 1 - \mu; \frac{1}{2}\right)$$

$$= 2^{i\tau-1/2}\Gamma(1-\mu)\left(\frac{\tau^2}{2}\right)^{\mu/2} J_{-\mu}(\tau\sqrt{2})[1+O(1/\tau)],\ \tau\to+\infty, \tag{1.162}$$

which reduces to (1.157).

Concerning the case of (1.156) in order to prove it we need to use formula (1.85) of analytic continuation of the Gauss function, because we have situation, when $x > 3$ or in particular $|1-x| > 2$. Therefore formula (1.85) gives the representation of the Gauss function in (1.56) as follows

$$\begin{aligned}
&{}_2F_1\left(1/2-i\tau, 1/2+i\tau; 1-\mu; \frac{1-x}{2}\right)\\
&= \frac{\Gamma(1-\mu)\Gamma(2i\tau)}{\Gamma(1/2+i\tau)\Gamma(1/2-\mu+i\tau)}\left(\frac{1-x}{2}\right)^{i\tau-1/2}\\
&\times {}_2F_1\left(1/2-i\tau, 1/2+\mu-i\tau; 1-2i\tau; \frac{2}{1-x}\right)\\
&+\frac{\Gamma(1-\mu)\Gamma(-2i\tau)}{\Gamma(1/2-i\tau)\Gamma(1/2-\mu-i\tau)}\left(\frac{1-x}{2}\right)^{-i\tau-1/2}\\
&\times {}_2F_1\left(1/2+i\tau, 1/2+\mu+i\tau; 1+2i\tau; \frac{2}{1-x}\right)\\
&= 2\Re_{i\tau}\left[\frac{\Gamma(1-\mu)\Gamma(2i\tau)}{\Gamma(1/2+i\tau)\Gamma(1/2-\mu+i\tau)}\left(\frac{1-x}{2}\right)^{i\tau-1/2}\right.\\
&\left.\times {}_2F_1\left(1/2-i\tau, 1/2+\mu-i\tau; 1-2i\tau; \frac{2}{1-x}\right)\right],
\end{aligned} \tag{1.163}$$

where we denote by the symbol $\Re_z\,[F(z)]$ (or respectively by $\Im_z\,[F(z)]$) the real (imaginary) part of function $F(z)$ by variable z. So it consists of two Gauss's hypergeometric functions and each of theirs can be treated similarly to case (1.155) involving the asymptotic of gamma-multipliers by using Stirling's formula (1.32). Thus we have the relation

$$\begin{aligned}
&\frac{\Gamma(2i\tau)}{\Gamma(1/2+i\tau)\Gamma(1/2-\mu+i\tau)}\\
&= e^{i\pi(\mu-1/2)/2}\frac{2^{2i\tau-1}}{\sqrt{\pi}}\tau^{\mu-1/2}[1+O(1/\tau)],\ \tau\to+\infty.
\end{aligned} \tag{1.164}$$

For the last Gauss function at the right-hand side of (1.163) we have the expansion of type

$$\begin{aligned}
&{}_2F_1\left(1/2-i\tau, 1/2+\mu-i\tau; 1-2i\tau; \frac{2}{1-x}\right)\\
&= \sum_{n=0}^{\infty}\frac{(1/2-i\tau)_n(1/2+\mu-i\tau)_n}{(1-2i\tau)_n n!}\left(\frac{2}{1-x}\right)^n\\
&= \sum_{n=0}^{\infty}\frac{(-i\tau)^n}{(1-x)^n n!}(1+O(1/\tau))
\end{aligned}$$

$$= \exp\left(\frac{i\tau}{x-1}\right)(1+O(1/\tau)),\ \tau \to +\infty. \tag{1.165}$$

Thus, finally substituting these expansions to (1.163) and invoking with (1.56) as well as an elementary Euler's formula we obtain the asymptotic equality (1.156). Theorem 1.8 is completely proved. •

Now we turn to write the asymptotic expansion for the square of the Macdonald function (1.98). In this case we use its representation through Meijer's G-function (1.117). Conversely this G-function can be expressed by the Slater theorem 1.6 through key function (1.69) as the combination of the hypergeometric ${}_1F_2$-functions. Namely, we find (using notation as in (1.163))

$$K_{i\tau}^2(x) = \frac{\Gamma(i\tau)\Gamma(-i\tau)}{2}{}_1F_2(1/2;\ 1+i\tau,\ 1-i\tau;\ x^2)$$

$$+\sqrt{\pi}\Re_{i\tau}\left[x^{-2i\tau}\frac{\Gamma(i\tau)\Gamma(2i\tau)}{\Gamma(1/2+i\tau)}{}_1F_2(1/2-i\tau;\ 1-i\tau,\ 1-2i\tau;\ x^2)\right],\ x>0. \tag{1.166}$$

Theorem 1.9. *For the square of the Macdonald function the following asymptotic expansion by its index takes place*

$$K_{i\tau}^2(x) = \left(\frac{\pi}{2\sinh(\pi\tau)\sqrt{\tau^2-x^2}} - \frac{\pi e^{-\pi\tau}}{\tau}\sin\left(2\tau - \frac{x^2}{2\tau} + 2\tau\log\left(\frac{x}{2\tau}\right)\right)\right)$$

$$\times\left[1+O(1/\tau)\right],\ \tau\to+\infty,\ 0<x<X,\ X>0. \tag{1.167}$$

Proof. We start from formula (1.166). As in previous theorem use the Stirling formula (1.146) to treat gamma-functions and Pochhammer symbols accounting the uniform convergence of the series. So we reduce expansion (1.166) for sufficiently large $\tau > 0$ to the following equality

$$K_{i\tau}^2(x) = \frac{\pi}{2\tau\sinh(\pi\tau)}\sum_{n=0}^{\infty}\frac{(1/2)_n}{(i\tau)^n(-i\tau)^n}\frac{x^n}{n!}[1+O(1/\tau)]$$

$$-\frac{\pi e^{-\pi\tau}}{\tau}\Im_{i\tau}\left[\exp\left(2i\tau\left(\log\frac{x}{2\tau}+1\right)\right)\sum_{n=0}^{\infty}\frac{(-ix^2/(2\tau))^n}{n!}\right][1+O(1/\tau)]$$

$$= \left(\frac{\pi}{2\sinh(\pi\tau)\sqrt{\tau^2-x^2}} - \frac{\pi e^{-\pi\tau}}{\tau}\sin\left(2\tau-\frac{x^2}{2\tau}+2\tau\log\frac{x}{2\tau}\right)\right)$$

$$\times\left[1+O(1/\tau)\right],\ 0<x<X,\ X>0, \tau\to+\infty. \tag{1.168}$$

Theorem 1.9 is proved. •

Let us consider several other particular cases of G-function (1.59) to establish their asymptotic expansions by index τ. For example, applying the Slater theorem to G-function (1.116) with $\nu = i\tau$ we have the following relation for the respective Bessel functions, namely

$$[I_{i\tau}(x) + I_{-i\tau}(x)]\, K_{i\tau}(x)$$
$$= \frac{\cosh(\pi\tau)}{\sqrt{\pi}} \Re_{i\tau} \left[\frac{\Gamma(-2i\tau)\Gamma(1/2+i\tau)}{\Gamma(1+i\tau)} x^{2i\tau} \right.$$
$$\left. \times {}_1F_2(1/2+i\tau;\ 1+i\tau,\ 1+2i\tau;\ x^2) \right]. \tag{1.169}$$

Similarly to previous asymptotic expansion (1.167) for the square of the Macdonald function it is not difficult to obtain the next theorem.

Theorem 1.10. *For the combination of Bessel functions* (1.169) *the following asymptotic expansion by index is true*

$$[I_{i\tau}(x) + I_{-i\tau}(x)]\, K_{i\tau}(x)$$
$$= \frac{1}{\tau} \cos\left(2\tau \log \frac{x}{2\tau} + 2\tau + \frac{x^2}{2\tau} \right) [1 + O(1/\tau)]\,,\ \tau \to +\infty, \tag{1.170}$$

and argument x belongs to some interval $(0,\ X],\ X > 0$.

Actually, using representation (1.169) by Stirling's formula one can reduce the hypergeometric series to elementary exponents and by Euler's trigonometric formula lead to asymptotic equality (1.170).

In order to investigate the asymptotic by the second index of the Whittaker function (1.131) put there $\sigma = i\tau$ and express it by the Slater theorem in terms of Kummer's function (1.133). As a result we arrive to the following equality

$$e^{-x/2} W_{\rho,i\tau}(x) = 2\Re_{i\tau} \left[\frac{\Gamma(-2i\tau)}{\Gamma(1/2-\rho-i\tau)} x^{i\tau+1/2} {}_1F_1(1/2+i\tau+\rho;\ 1+2i\tau;\ -x) \right],$$
$$0 < x < X,\ X > 0. \tag{1.171}$$

The proof of the next theorem is absolutely analogously to previous ones and can be reestablished by the reader without difficulties. Nevertheless, we demonstrate it briefly here.

Theorem 1.11. *Under condition* $x \in (0,\ X],\ X > 0\ \rho \in \mathbf{R}$ *for the Whittaker function* (1.131) *the asymptotic expansion by index* τ *holds*

$$W_{\rho,i\tau}(x) = \sqrt{2x} e^{-\pi\tau/2} \tau^{\rho-1/2}$$
$$\times \cos\left(\tau \log \frac{x}{4\tau} - \frac{\pi}{2}(\rho - 1/2) + \tau \right) [1 + O(1/\tau)]\,, \tau \to +\infty. \tag{1.172}$$

Proof. Indeed, turning to representation (1.171) write the series for the Kummer function ${}_1F_1(1/2+i\tau+\rho;\ 1+2i\tau;-x)$ owing to formula (1.45) as follows

$$ {}_1F_1(1/2+i\tau+\rho;\ 1+2i\tau;-x)=\lim_{M\to\infty}\sum_{n=0}^{M}\frac{(1/2+i\tau+\rho)_n}{(1+2i\tau)_n}\frac{(-x)^n}{n!} $$

$$ =\lim_{M\to\infty}\sum_{n=0}^{M}\frac{(i\tau)^n}{(2i\tau)^n}\frac{(-x)^n}{n!}[1+O(1/\tau)],\ \ \tau\to+\infty, x\in(0,X]. $$

Consequently, we obtain the asymptotic expansion of the Kummer function by index τ as

$$ {}_1F_1(1/2+i\tau+\rho;\ 1+2i\tau;-x)=e^{-x/2}[1+O(1/\tau)],\ \ \tau\to+\infty, x\in(0,X], \rho\in\mathbf{R}. $$

Meanwhile, the gamma-ratio in (1.171) can be treated by Stirling's formula (1.32) and we arrive to the expansion

$$ \frac{\Gamma(-2i\tau)}{\Gamma(1/2-\rho-i\tau)}=\frac{1}{\sqrt{2}}\tau^{\rho-1/2}e^{-\pi\tau/2}\exp\left(i\left(\tau-\frac{\pi}{2}(\rho-1/2)+\tau\log\frac{1}{4\tau}\right)\right) $$
$$ \times[1+O(1/\tau)],\ \ \tau\in+\infty. $$

Hence, substituting these expressions into equality (1.171) and appealing to the elementary Euler trigonometric formula lead finally to (1.172). Theorem 1.11 is proved. ●

It is interesting to consider now special case of the Gauss hypergeometric function (1.47) in view of its asymptotic by isolated parameter $i\tau$. Indeed, setting at (1.47) $a=\xi-i\tau,\ b=\xi+i\tau,\ c=1+\xi-\sigma$, where $\xi,\ \sigma$ are some real parameters we have more general result as Theorem 1.8, namely

Theorem 1.12. *For Gauss's hypergeometric function the next asymptotic expansions by its index τ take place*

$$ {}_2F_1(\xi-i\tau,\xi+i\tau;1+\xi-\sigma;-x)=\Gamma(1+\xi-\sigma)\tau^{\sigma-\xi}x^{(\sigma-\xi)/2} $$
$$ \times J_{\xi-\sigma}(2\tau\sqrt{x})\left[1+O(1/\tau)\right],\ \ 0<x<1,\ \ \tau\to+\infty; \tag{1.173} $$
$$ {}_2F_1(\xi-i\tau,\xi+i\tau;1+\xi-\sigma;-1)=\Gamma(1+\xi-\sigma)\tau^{\sigma-\xi}2^{i\tau-(\sigma+3\xi)/2} $$
$$ \times J_{\xi-\sigma}(\tau\sqrt{2})\left[1+O(1/\tau)\right],\ \ \tau\to+\infty,\ \ \xi+\sigma<2; \tag{1.174} $$
$$ {}_2F_1(\xi-i\tau,\xi+i\tau;1+\xi-\sigma;-x)=\frac{\Gamma(1+\xi-\sigma)}{\sqrt{\pi}\tau^{\xi-\sigma+1/2}x^{\xi}} $$
$$ \times\cos\left(\tau\log(4x)-(\xi+1/2-\sigma)\pi/2+\frac{\tau}{2x}\right)\left[1+O(1/\tau)\right],\ \ \tau\to+\infty;\ \ x>1. \tag{1.175} $$

Finally we attract our attention to investigate the asymptotic of the Meijer G-function (1.60) by two isolated parameters. Indeed, let us take the following G-function

$$ G^{m+2,n}_{p,q+2}\left(x\left|\begin{matrix}(\alpha_p)\\ i\tau,\ -i\tau,\ (\beta_q)\end{matrix}\right.\right),\ \ x>0, \tag{1.176} $$

provided that parameters satisfy conditions described above. First of all consider key functions (1.69), (1.71) related to G-function (1.176). Namely, invoking with the Gauss-Legendre formula (1.30) functions (1.69), (1.71) take forms

$$\sum\nolimits_m^{i\tau}(x) = \frac{1}{\sqrt{\pi}}\Re_{i\tau}\left[(4x)^{i\tau}\Gamma\left[\begin{array}{l} i\tau,\ 1/2+i\tau,\ (\beta_m)+i\tau,\ 1-(\alpha_n)-i\tau \\ (\alpha_p^{n+1})+i\tau,\ 1-(\beta_q^{m+1})-i\tau \end{array}\right]\right.$$

$$\left.\times{}_pF_{q+1}\left[\begin{array}{l} 1-(\alpha_n)-i\tau,\ 1-(\alpha_p^{n+1})-i\tau; \\ 1-2i\tau,\ 1-(\beta_m)-i\tau,\ 1-(\beta_q^{m+1})-i\tau \end{array} (-1)^{p-n-m}x\right]\right]$$

$$+\sum_{j=1}^{m} x^{\beta_j}\Gamma\left[\begin{array}{l} i\tau-\beta_j,\ -i\tau-\beta_j, (\beta_m)'-\beta_j,\ 1-(\alpha_n)+\beta_j \\ (\alpha_p^{n+1})-\beta_j,\ 1-(\beta_q^{m+1})+\beta_j \end{array}\right]$$

$$\times{}_pF_{q+1}$$

$$\left[\begin{array}{l} 1-(\alpha_n)+\beta_j,\ 1-(\alpha_p^{n+1})+\beta_j; \\ 1-i\tau+\beta_j,\ 1+i\tau+\beta_j, 1-(\beta_m)'+\beta_j, 1-(\beta_q^{m+1})+\beta_j \end{array} (-1)^{p-n-m}x\right], \quad (1.177)$$

$$\sum\nolimits_n^{i\tau}(1/x) = \sum_{j=1}^{n} x^{\alpha_j-1}\Gamma\left[\begin{array}{l} \alpha_j-(\alpha_n)', 1-\alpha_j+(\beta_m),\ 1-\alpha_j+i\tau,\ 1-\alpha_j-i\tau \\ \alpha_j-(\beta_q^{m+1}),\ 1-\alpha_j+(\alpha_p^{n+1}) \end{array}\right]$$

$$\times{}_{q+2}F_{p-1}$$

$$\left[\begin{array}{l} 1-\alpha_j+i\tau,\ 1-\alpha_j-i\tau,\ 1-\alpha_j+(\beta_m), 1-\alpha_j+(\beta_q^{m+1});\ \frac{(-1)^{q-n-m}}{x} \\ 1-\alpha_j+(\alpha_n)',\ 1-\alpha_j+(\alpha_p^{n+1}) \end{array}\right]. \quad (1.178)$$

Hence, assuming that all conditions of the Slater Theorem 1.6 are fulfilled for G-function (1.176) one can define it as in formulae (1.78)- (1.81), precisely

$$G^{m+2,n}_{p,q+2}\left(x\left|\begin{array}{c}(\alpha_p)\\ i\tau,\ -i\tau,\ (\beta_q)\end{array}\right.\right) = \sum\nolimits_m^{i\tau}(x), \quad 0<x<X,\ X>1 \text{ if } q+2>p, \quad (1.179)$$

$$G^{m+2,n}_{p,q+2}\left(x\left|\begin{array}{c}(\alpha_p)\\ i\tau,\ -i\tau,\ (\beta_q)\end{array}\right.\right) = \sum\nolimits_m^{i\tau}(x), \quad 0<x<1, \text{ if } q+2=p, \quad (1.180)$$

$$G^{m+2,n}_{p,q+2}\left(x\left|\begin{array}{c}(\alpha_p)\\ i\tau,\ -i\tau,\ (\beta_q)\end{array}\right.\right) = \sum\nolimits_n^{i\tau}(1/x), \quad 1<x<X, \text{ if } q+2=p, \quad (1.181)$$

$$G^{m+2,n}_{p,q+2}\left(x\left|\begin{array}{c}(\alpha_p)\\ i\tau,\ -i\tau,\ (\beta_q)\end{array}\right.\right) = \sum\nolimits_n^{i\tau}(1/x), \quad 0<x<X, \text{ if } q+2<p. \quad (1.182)$$

Now our purpose is to write asymptotic expansions of sums (1.177), (1.178). For this use formula (1.146) and its analog for $\Gamma(\alpha-i\tau)$ which can be easily derived from the Stirling formula (1.32). Let us consider sum (1.177). We have for sufficiently large positive τ that

$$\Gamma\left[\begin{array}{l} i\tau,\ 1/2+i\tau,\ (\beta_m)+i\tau, 1-(\alpha_n)-i\tau \\ (\alpha_p^{n+1})+i\tau, 1-(\beta_q^{m+1})-i\tau \end{array}\right]$$

$$= (2\pi)^{m+n+1-\frac{p+q}{2}}e^{-\left(m+n+1-\frac{p+q}{2}\right)\pi\tau}\tau^{\ varrho-1/2}$$

$$\times\exp\left[i\left((2+q-p)(\tau\log\tau-\tau)+\frac{\pi}{2}\chi\right)\right](1+O(1/\tau)),\ \tau\to+\infty, \quad (1.183)$$

where

$$\chi = \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j + \sum_{j=1}^{n} \alpha_j - \sum_{j=n+1}^{p} \alpha_j - m - n + \frac{p+q-1}{2} \tag{1.184}$$

and parameter ϱ is defined by formula (1.76). As we presupposed above that all free parameters are real numbers to omit their unessential imaginary parts at asymptotic expansions. Further one can write that

$$ {}_pF_{q+1}\left[\begin{matrix} 1-(\alpha_n)-i\tau,\ 1-(\alpha_p^{n+1})-i\tau; \\ 1-2i\tau,\ 1-(\beta_m)-i\tau,\ 1-(\beta_q^{m+1})-i\tau \end{matrix} (-1)^{p-n-m}x\right]$$

$$= \lim_{M\to\infty} \sum_{k=0}^{M} \frac{(1-(\alpha_n)-i\tau)_k(1-(\alpha_p^{n+1})-i\tau)_k}{(1-2i\tau)_k(1-(\beta_m)-i\tau)_k(1-(\beta_q^{m+1})-i\tau)_k} \frac{((-1)^{p-n-m}x)^k}{k!}$$

$$= \lim_{M\to\infty} \sum_{k=0}^{M} \frac{\left((-1)^{m+n+q+1}x(i\tau)^{p-q-1}\right)^k}{k!}(1+O(1/\tau))$$

$$= \exp\left[(-1)^{m+n+q+1}x(i\tau)^{p-q-1}\right](1+O(1/\tau)),\ \tau\to+\infty. \tag{1.185}$$

As it is easily seen accounting the asymptotic expansion of the product of Pochhammer symbols like (1.159) the last sum in (1.177) for sufficiently large positive τ can be changed by the following expression, namely

$$\sum_{j=1}^{m} x^{\beta_j}\Gamma\left[\begin{matrix} i\tau-\beta_j,\ -i\tau-\beta_j, (\beta_m)'-\beta_j,\ 1-(\alpha_n)+\beta_j \\ (\alpha_p^{n+1})-\beta_j,\ 1-(\beta_q^{m+1})+\beta_j \end{matrix}\right]$$

$$\times {}_pF_{q+1}$$

$$\left[\begin{matrix} 1-(\alpha_n)+\beta_j,\ 1-(\alpha_p^{n+1})+\beta_j; \\ 1-i\tau+\beta_j,\ 1+i\tau+\beta_j, 1-(\beta_m)'+\beta_j, 1-(\beta_q^{m+1})+\beta_j \end{matrix} (-1)^{p-n-m}x\right]$$

$$= 2\pi\frac{e^{-\pi\tau}}{\tau}\sum_{j=1}^{m}\left(\frac{x}{\tau^2}\right)^{\beta_j}\Gamma\left[\begin{matrix} (\beta_m)'-\beta_j,\ 1-(\alpha_n)+\beta_j \\ (\alpha_p^{n+1})-\beta_j,\ 1-(\beta_q^{m+1})+\beta_j \end{matrix}\right]$$

$$\times {}_pF_{q-1}\left[\begin{matrix} 1-(\alpha_n)+\beta_j,\ 1-(\alpha_p^{n+1})+\beta_j; \\ 1-(\beta_m)'+\beta_j, 1-(\beta_q^{m+1})+\beta_j \end{matrix} \frac{(-1)^{p-n-m}x}{\tau^2}\right]$$

$$\times(1+O(1/\tau)),\ \tau\to+\infty. \tag{1.186}$$

It is clear from relation (1.185) that we may consider two cases of asymptotic of sum (1.177). Indeed, if $p-q=2l+1,\ l=0,\ \pm1,\ \pm2,\dots$, then one can deduce the final asymptotic formula as

$$\sum\nolimits_{m}^{i\tau}(x) = \frac{(2\pi)^{m+n+1-\frac{p+q}{2}}}{\sqrt{\pi}}e^{-\left(m+n+1-\frac{p+q}{2}\right)\pi\tau}$$

$$\times\tau^{\varrho-1/2}\exp\left[(-1)^{m+n+q+l+1}x\tau^{2l}\right]$$

$$\times\cos\left((2+q-p)(\tau\log\tau-\tau)+\tau\log(4x)+\frac{\pi}{2}\chi\right)$$

$$+2\pi\frac{e^{-\pi\tau}}{\tau}\sum_{j=1}^{m}\left(\frac{x}{\tau^2}\right)^{\beta_j}\Gamma\left[\begin{array}{l}(\beta_m)'-\beta_j,\ 1-(\alpha_n)+\beta_j\\ (\alpha_p^{n+1})-\beta_j,\ 1-(\beta_q^{m+1})+\beta_j\end{array}\right]$$

$$\times{}_pF_{q-1}\left[\begin{array}{l}1-(\alpha_n)+\beta_j,\ 1-(\alpha_p^{n+1})+\beta_j;\\ 1-(\beta_m)'+\beta_j, 1-(\beta_q^{m+1})+\beta_j\end{array}\ \frac{(-1)^{p-n-m}x}{\tau^2}\right]$$

$$\times(1+O(1/\tau)),\ \tau\to+\infty. \tag{1.187}$$

For $p-q=2l,\ l=\pm1,\ \pm2,\ \dots$, we obtain

$$\sum{}_m^{i\tau}(x)=\frac{(2\pi)^{m+n+1-\frac{p+q}{2}}}{\sqrt{\pi}}e^{-\left(m+n+1-\frac{p+q}{2}\right)\pi\tau}\tau^{\varrho-1/2}$$

$$\times\cos\left((2+q-p)(\tau\log\tau-\tau)+\tau\log(4x)+(-1)^{m+n+q+l}x\tau^{2l-1}+\frac{\pi}{2}\chi\right)$$

$$+2\pi\frac{e^{-\pi\tau}}{\tau}\sum_{j=1}^{m}\left(\frac{x}{\tau^2}\right)^{\beta_j}\Gamma\left[\begin{array}{l}(\beta_m)'-\beta_j,\ 1-(\alpha_n)+\beta_j\\ (\alpha_p^{n+1})-\beta_j,\ 1-(\beta_q^{m+1})+\beta_j\end{array}\right]$$

$$\times{}_pF_{q-1}\left[\begin{array}{l}1-(\alpha_n)+\beta_j,\ 1-(\alpha_p^{n+1})+\beta_j;\\ 1-(\beta_m)'+\beta_j, 1-(\beta_q^{m+1})+\beta_j\end{array}\ \frac{(-1)^{p-n-m}x}{\tau^2}\right]$$

$$\times(1+O(1/\tau)),\ \tau\to+\infty. \tag{1.188}$$

By the same treatment for sum (1.178) immediately arrive to the asymptotic expansion by index τ of type

$$\sum{}_n^{i\tau}(1/x)=2\pi\frac{\tau}{x}e^{-\pi\tau}\sum_{j=1}^{n}\left(\frac{x}{\tau^2}\right)^{\alpha_j}\Gamma\left[\begin{array}{l}\alpha_j-(\alpha_n)', 1-\alpha_j+(\beta_m)\\ \alpha_j-(\beta_q^{m+1}),\ 1-\alpha_j+(\alpha_p^{n+1})\end{array}\right]$$

$$\times{}_qF_{p-1}\left[\begin{array}{l}1-\alpha_j+(\beta_m), 1-\alpha_j+(\beta_q^{m+1});\\ 1-\alpha_j+(\alpha_n)',\ 1-\alpha_j+(\alpha_p^{n+1})\end{array}\ \frac{(-1)^{q-n-m}\tau^2}{x}\right]$$

$$\times(1+O(1/\tau)),\ \tau\to+\infty. \tag{1.189}$$

Thus we obtained the following theorem.

Theorem 1.13. *For the Meijer G-function* (1.176) *with real free parameters* (α_p), (β_q) *and argument* $x\in(0,X]$, $X>1$ *asymptotic expansions* (1.187) – (1.189) *by its index* $\tau\to+\infty$ *are true provided that the respective sums* (1.177) – (1.178) *are chosen according to Slater's theorem described by formulae* (1.179) – (1.182).

1.3 Fourier's, Mellin's and Laplace's transforms

In this section we give briefly some auxiliary results from classical one-dimensional integral transforms theory, namely some definitions and theorems concerning the Fourier, the Mellin and the Laplace integral transforms. Details and proofs of the theorems the reader can find in Titchmarsh [1]. These results are repeatedly used below to establish the respective properties for index integral transforms.

As it is known, the classical one-dimensional integral transforms are of the form

$$[Kf](x) = \int_{-\infty}^{+\infty} K(x,t)f(t)dt, \ x \in \mathbf{R}, \tag{1.190}$$

where $K(x,t)$ is some given function (kernel of the transform), $f(t)$ is an original in a certain space of functions, and $[Kf](x)$ is the image of the function $f(t)$. Undoubtedly, among the most important classical integral transforms one is *the Fourier transform* with $K(x,t) = e^{ixt}$, precisely

$$[Ff](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ixt} f(t)dt, \ x \in \mathbf{R}. \tag{1.191}$$

According to discussions at the beginning of this chapter one can imply the convergence of the integral (1.191) in different meaning. In this book we attract mostly our attention to L_p-theorems for integral transforms. In particular, for the Fourier transform (1.191) we formulate the theorem from Titchmarsh [1] in terms of $L_p(\mathbf{R})$-convergence like, for example (1.6).

Theorem 1.14. *Let $f(x)$ belong to $L_p(\mathbf{R})$, where $1 < p \leq 2$. Then, as $E \to \infty$,*

$$[Ff](x,E) = \frac{1}{\sqrt{2\pi}} \int_{-E}^{E} e^{ixt} f(t)dt \tag{1.192}$$

converges in mean with exponent $q = p/(p-1)$. The mean limit $[Ff](x)$, called the Fourier transform of $f(x)$, satisfies

$$\int_{-\infty}^{+\infty} |[Ff](t)|^q dt \leq \frac{1}{(2\pi)^{\frac{q}{2}-1}} \int_{-\infty}^{+\infty} |f(t)|^p dt. \tag{1.193}$$

The Fourier reciprocity holds in the sense that

$$[Ff](x) = \frac{1}{\sqrt{2\pi}} \frac{d}{dx} \int_{-\infty}^{+\infty} \frac{e^{ixt}-1}{it} f(t)dt, \tag{1.194}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \frac{d}{dx} \int_{-\infty}^{+\infty} \frac{1-e^{-ixt}}{it} [Ff](t)dt, \tag{1.195}$$

almost everywhere on $\mathbf{R}$.

Let us note that we might replace $[Ff](x, E)$ by

$$[Ff](x, E, M) = \frac{1}{\sqrt{2\pi}} \int_{-E}^{M} e^{ixt} f(t) dt, \tag{1.196}$$

where $E \to \infty$, $M \to \infty$, in any manner.

Letting at Theorem 1.14 $p = 2$, we obtain known *Plancherel's theorem* and inequality (1.193) becomes *the Parseval equality* for the Fourier transform (see details in Titchmarsh [1]).

Using elementary Euler's trigonometric formula $e^{ixt} = \cos xt + i \sin xt$ one can introduce and formulate the same theorem for *the cosine- and the sine-Fourier transforms*, namely

$$[F_c f](x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos xt dt, \tag{1.197}$$

$$[F_s f](x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin xt dt, \tag{1.198}$$

where we consider these transforms in spaces $L_p(\mathbf{R}_+)$, $1 \le p \le 2$. As it is well known these transforms have symmetric inversion formulae, namely almost everywhere on R_+ the following reciprocities or dual formulae hold

$$[F_c f](x) = \sqrt{\frac{2}{\pi}} \frac{d}{dx} \int_0^\infty f(t) \frac{\sin xt}{t} dt, \tag{1.199}$$

$$f(x) = \sqrt{\frac{2}{\pi}} \frac{d}{dx} \int_0^\infty [F_c f](t) \frac{\sin xt}{t} dt, \tag{1.200}$$

$$[F_s f](x) = \sqrt{\frac{2}{\pi}} \frac{d}{dx} \int_0^\infty f(t) \frac{1 - \cos xt}{t} dt, \tag{1.201}$$

$$f(x) = \sqrt{\frac{2}{\pi}} \frac{d}{dx} \int_0^\infty [F_s f](t) \frac{1 - \cos xt}{t} dt. \tag{1.202}$$

In spite of the fact, that the Fourier transform (1.191) is the most important one we may consider separately *the Mellin transform*, which can be obtained from the Fourier transform by an exponential replacement and by rotating the complex plane through a right angle

$$f^*(s) = \mathcal{M}\{f(t); s\} = [F\sqrt{2\pi} f(e^y)](-is) = \int_0^{+\infty} f(t) t^{s-1} dt. \tag{1.203}$$

In fact, its inversion is given by the formula

$$f(x) = \mathcal{M}^{-1}\{f^*(s); x\} = \frac{1}{2\pi i} \int_{\nu - i\infty}^{\nu + i\infty} f^*(s) x^{-s} ds, \ \nu = \Re s, \ x > 0, \tag{1.204}$$

and undoubtedly, we can observe its connection with the class of hypergeometric type functions as it was described in detail in Marichev [1]. Definitions of the Meijer G-function (1.59), the Fox H-function (1.63) and wide list of their particular cases can be interpreted in terms of the Mellin transform formulae (1.203)-(1.204). Here if we denote by $\to$ the correspondence between a function and its Mellin's transform, then the following formulae of general type can be easily proved

$$f(ax) \to a^{-s}f^*(s),\ a > 0; \tag{1.205}$$

$$x^p f(x) \to f^*(s+p); \tag{1.206}$$

$$f(x^p) \to \frac{1}{|p|}f^*(s/p),\ p \neq 0; \tag{1.207}$$

$$f^{(n)}(x) \to \frac{\Gamma(n+1-s)}{\Gamma(1-s)}f^*(s-n),\ \lim_{x\to 0} x^{s-k-1}f^{(k)}(x) = 0,$$
$$k = 0, 1, \ldots, n-1; \tag{1.208}$$

$$\left(x\frac{d}{dx}\right)^n f(x) \to (-s)^n f^*(s); \tag{1.209}$$

$$\left(\frac{d}{dx}x\right)^n f(x) \to (1-s)^n f^*(s). \tag{1.210}$$

The corresponding L_p-theorems for the Mellin transform (1.203) and its inversion (1.204) are more suitable formulated in the weighted $L_{\nu,p}$-spaces (1.19). According to the definition of the Mellin transform these results can be easily obtained from the Fourier transform theory (see details in Titchmarsh [1]).

Theorem 1.15. *Let $f(x)$ belong to the space $L_{\nu,p}(\mathbf{R}_+)$, $1 < p \leq 2$, $\nu \in \mathbf{R}$. Then its Mellin's transform* (1.203) *$f^*(s) \equiv f^*(\nu+it)$ exists and belongs to the space $L_q(\mathbf{R})$. Moreover, the convergence of the integral* (1.203) *is in mean with exponent q by the norm of the space $L_q(\nu - i\infty, \nu + i\infty)$, namely*

$$||f^*(s) - \int_{1/N}^{N} f(t)t^{s-1}dt||_{L_q(\nu-i\infty,\ \nu+i\infty)} \to 0,\ N \to \infty, \tag{1.211}$$

where

$$||f^*||_{L_q(\nu-i\infty,\ \nu+i\infty)} = \frac{1}{2\pi}\left(\int_{\nu-i\infty}^{\nu+i\infty} |f(s)ds|^q\right)^{1/q}. \tag{1.212}$$

Theorem 1.16. *Let $f^*(\nu+it) \in L_p(\mathbf{R})$, $1 < p \leq 2$, $\nu \in \mathbf{R}$. Then the inverse Mellin transform* (1.204) *exists and belongs to the space $L_{\nu,q}(\mathbf{R}_+)$. Moreover, the convergence of integral* (1.204) *is in mean with exponent q by the norm of the space $L_{\nu,q}(\mathbf{R}_+)$ and almost everywhere on $\mathbf{R}_+$ the dual equality is true*

$$f(x) = \frac{1}{2\pi i}\frac{d}{dx}\int_{\nu-i\infty}^{\nu+i\infty}\frac{f^*(s)}{1-s}x^{1-s}ds,\ x > 0. \tag{1.213}$$

Theorem 1.17. *Let* $f^*(\nu+it) \in L_p(\mathbf{R}),\ 1 < p \le 2,\ \nu \in \mathbf{R},\ h(x) \in L_{1-\nu,p}(\mathbf{R}_+)$. *Then the Mellin-Parseval equality takes place*

$$\int_0^\infty f(xt)h(t)dt = \frac{1}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty} f^*(s)h^*(1-s)x^{-s}ds,\ x > 0, \tag{1.214}$$

where $h^*(s)$ *is the Mellin transform* (1.203) *of the function* $h(x)$.

To introduce *the Laplace transform* for positive variable x one can call just demonstrated equality (1.214). Indeed, letting there $f(x) = e^{-x}$ we find, that the left-hand side of (1.214) becomes the Laplace transform like (1.190) and usually we shall define it as

$$[Lf](x) = \int_0^\infty e^{-xt}f(t)dt,\ x > 0. \tag{1.215}$$

From Euler's integral (1.22) observe that the Mellin transform (1.203) of exponent function e^{-x} is the gamma-function $\Gamma(s)$ and Stirling's formula (1.32) allows us to conclude that $\Gamma(\nu+it) \in L_p(\mathbf{R}),\ 1 < p \le 2,\ \nu \in \mathbf{R}_+$. Therefore, if $f(t) \in L_{1-\nu,p}(\mathbf{R}_+)$, then we have the following equality from (1.214) as

$$[Lf](x) = \frac{1}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty} \Gamma(s)f^*(1-s)x^{-s}ds,\ x > 0. \tag{1.216}$$

As is known (see Titchmarsh [1], Marichev [1], Yakubovich and Luchko [2]) the left-hand side of (1.214) is slightly different from *the Mellin convolution*

$$(f * g)(x) = \int_0^\infty f\left(\frac{x}{t}\right)g(t)\frac{dt}{t} \tag{1.217}$$

and we can definitely call the Laplace transform of positive variable x as the Mellin convolution type integral transform as noted in Widder [1], Hirschman and Widder [1], Marichev [1], Brychkov et al. [1, 1983]. In the next section we shall consider some properties of such transforms.

Let us demonstrate the known *Post-Widder* inversion formula for the Laplace transform (1.215) (see details in Widder [1])

$$f(t) = \lim_{k\to\infty}\frac{(-1)^k}{k!}[Lf]^{(k)}\left(\frac{k}{t}\right)\left(\frac{k}{t}\right)^{k+1}. \tag{1.218}$$

This formula involves derivatives of the Laplace transform $[Lf](x)$ and it is clear that one needs to know it only for large positive values of x. Furthermore, (see in Widder [1]) the range of the Laplace transform is completely described for L_p-functions, namely arbitrary function $g(x)$ is the Laplace transform (1.215) of some function $f(t) \in L_p(\mathbf{R}_+),\ 1 < p < \infty$ if and only if g is infinitely differentiable and, for some constant M,

$$\frac{k}{(k!)^p}\int_0^\infty \left|\frac{d^kg}{dx^k}\right|^k x^{kp+p-2}dx < M,\ k = 0,\ 1,\ 2,\dots. \tag{1.219}$$

1.4 General Mellin convolution type integral transforms

In this section we collect some results concerning properties of general integral transforms of convolution type of (1.214), (1.217). We note the basic works on this matter as Titchmarsh [1], Widder [1], Hirschman and Widder [1], Watson [1], Srivastava and Buschman [1], Samko et al. [1], Dzrbasjan [1], Rooney [1], [2], Fox [1], Marichev [1], Nguyen Thanh Hai and Yakubovich [1], Yakubovich and Luchko [2].

We discuss here the following Mellin convolution type transforms

$$[Kf](x) = \int_0^\infty k(xy)f(y)dy, \ x > 0, \tag{1.220}$$

$$[\hat{K}f](x) = \int_0^\infty k\left(\frac{x}{y}\right) f(y)\frac{dy}{y}, \ x > 0. \tag{1.221}$$

In order to describe the range of these transforms we need to remind one remarkable theorem from Titchmarsh [1] on *the Fourier convolution* of two L_p-functions. After that one can appeal to this theorem following Rooney [2] to formulate the corresponding theorems for Mellin convolution transforms.

Theorem 1.18. *Let $f(x)$, $[Ff](x)$ be transforms of $L_p(\mathbf{R})$, $L_q(\mathbf{R})$, and $g(x)$, $[Fg](x)$ of $L_r(\mathbf{R})$, $L_{r'}(\mathbf{R})$, $r' = r/(r-1)$, where $p^{-1} + r^{-1} \geq 1$. Then the product of Fourier transforms $[Ff](x)[Fg](x)$ and the Fourier convolution*

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(y)g(x-y)dy \tag{1.222}$$

are Fourier transforms (1.191) *of classes $L_{P'}(\mathbf{R})$, $L_P(\mathbf{R})$ respectively, where*

$$P = \frac{pr}{p+r-pr}, \ P' = P/(P-1). \tag{1.223}$$

From the definition of the Mellin transform (1.203) we see that if $s = \nu + ix$, where $\nu \in \mathbf{R}$, $x \in \mathbf{R}$, then we obtain

$$f^*(\nu + ix) = [F\sqrt{2\pi}f(e^y)e^{\nu y}](x) = \int_0^{+\infty} f(t)t^{\nu+ix-1}dt. \tag{1.224}$$

Hence, it is clear that for the case of the Mellin transform is more suitable to use the weighted Lebesgue spaces $L_{\nu,p}(\mathbf{R}_+)$ with norm (1.19). Namely, for the Mellin convolution type transforms (1.220), (1.221) Theorem 1.18 gives respectively, (see Rooney [2])

Theorem 1.19. *Let $f(x) \in L_{\nu,p}(\mathbf{R}_+)$, $k(x) \in L_{1-\nu,r}(\mathbf{R}_+)$, where $1 \leq p \leq \infty$, $1 \leq r \leq \infty$ and $p^{-1} + r^{-1} \geq 1$. Then the transform* (1.220) *exists for almost all $x \in \mathbf{R}_+$*

and $[Kf]$ is the operator from the space $L_{\nu,p}(\mathbf{R}_+)$ into the space $L_{1-\nu,P}(\mathbf{R}_+)$, where parameter P is defined by equality (1.223).

Theorem 1.20. *Let $f(x) \in L_{\nu,p}(\mathbf{R}_+)$, $k(x) \in L_{\nu,r}(\mathbf{R}_+)$, where $1 \le p \le \infty$, $1 \le r \le \infty$ and $p^{-1} + r^{-1} \ge 1$. Then the transform* (1.221) *exists for almost all $x \in \mathbf{R}_+$ and $[\hat{K}f]$ is operator from the space $L_{\nu,p}(\mathbf{R}_+)$ into the space $L_{\nu,P}(\mathbf{R}_+)$, where parameter P is defined by equality* (1.223).

There are many works, which are devoted to study of various particular cases of the Mellin convolution type transforms. We already announced some of theirs above. Note here, that inversions of these transforms can be established using the Mellin-Parseval equality (1.214) (see details in Yakubovich and Luchko [2]). There is also the approach to define these transforms by the right-hand side of equality (1.214), namely so-called G- and H-transforms described in Vu Kim Tuan et al. [1], Vu Kim Tuan [3], Yakubovich and Luchko [2], Samko et al. [1]. These general transforms with hypergeometric type of special functions comprise a wide set of examples. We defined some of theirs as the cosine and the sine Fourier transforms (1.197), (1.198). Moreover, one can use the table of G-functions from Section 1.2 as the kernels of such transforms. The general theory of the Mellin convolution type transforms with so-called Watson or Fourier kernels from Titchmarsh [1], Watson [1] and Yakubovich and Luchko [2] is also worth mentioning on this matter. In addition, for our further purposes it is important to define here *the Hankel transform* as

$$[J_\mu f](x) = \int_0^\infty \sqrt{xy} J_\mu(xy) f(y) dy, \ \mu > -1 \tag{1.225}$$

with the Bessel function of the first kind (1.88). The boundedness of the Hankel transform (1.225) is given, for example by the following statement (see Rooney [2]).

Theorem 1.21. *If $1 < p < \infty$ and $\gamma(p) \le \nu < \Re\mu + 3/2$, where*

$$\gamma(p) = \max\left[\frac{1}{p}, \frac{1}{q}\right], \ q = p/(p-1),$$

then the operator $[J_\mu f](x)$ given by (1.225) *is bounded from $L_{\nu,p}(\mathbf{R}_+)$ to $L_{1-\nu,P}(\mathbf{R}_+)$ and is a one-to-one transform on $L_{\nu,p}(\mathbf{R}_+)$ for all $P \ge p$ such that $P' \ge \nu^{-1}$ and $1/P + 1/P' = 1$. For $1 < p \le 2$ and $f(x) \in L_{\nu,p}(\mathbf{R}_+)$*

$$\mathcal{M}\{[J_\mu f](t); s\} = 2^{s-1/2}\frac{\Gamma((\mu + s + 1/2)/2)}{\Gamma((\mu - s + 3/2)/2)}\mathcal{M}\{f(t); s\}, \ \Re s = 1 - \nu. \tag{1.226}$$

Let us remark finally that Mellin convolution type transforms contain the respective kernels $K(x, y)$ as the function of one variable $z = xy$ or $z = x/y$. This is fundamental difference of the considered convolution transforms from so-called index transforms,which we shall announce in the next section.

1.5 Notion of the index transforms

This book was intended as a first attempt to emphasize a special class of integral transforms whose kernels involve special functions of hypergeometric type with isolated parameters, so-called *index transforms*. As is shown throughout of this book the integration in inversion formulae of these transforms is realized with respect to the index and not the arguments as in the preceding examples of convolution transforms. Although we shall use very extensively the theory of convolution transforms and general integral operators below to construct the index transforms theory in L_p-spaces. More precisely, we need to describe their properties, to prove inversion formulae and to comprise various examples of such transforms. Furthermore, we shall consider so-called *index-convolution* integral transforms as mappings from one- to two-dimensional functional spaces. The choice of the hypergeometric approach (see details in Yakubovich and Luchko [2]) seems to be the best adapted to our theory.

The index transforms first appear from integral representations of arbitrary functions of Fourier type integrals (see Titchmarsh [1]) under respective conditions. We appeal to Erdélyi et al. [1] to demonstrate such known expansions with Bessel and Legendre functions as the kernels, namely

$$f(x) = -\frac{1}{2}\int_{-i\infty}^{i\infty} tJ_t(x)dt \int_0^\infty H_t^{(2)}(y)f(y)\frac{dy}{y}, \tag{1.227}$$

where

$$H_\nu^{(2)}(x) = J_\nu(x) - iY_\nu(x), \tag{1.228}$$

(Kontorovich and Lebedev [1]);

$$f(x) = \frac{1}{\pi^2}\int_{-\infty}^\infty e^{\frac{\pi(x+t)}{2}} K_{i(x+t)}(a)dt \int_{-\infty}^\infty e^{\frac{\pi(y+t)}{2}} K_{i(y+t)}(a)f(y)dy, \ a > 0, \tag{1.229}$$

(Crum [1]);

$$f(x) = \frac{1}{2}\int_0^\infty \frac{J_{it}(e^x) + J_{-it}(e^x)}{\sinh \pi t} tdt \int_{-\infty}^\infty [J_{it}(e^y) + J_{-it}(e^y)]f(y)dy, \tag{1.230}$$

(Titchmarsh [2]);

$$xf(x) = \frac{2}{\pi^2}\int_0^\infty t\sinh \pi t K_{it}(x)dt \int_0^\infty K_{it}(y)f(y)dy, \tag{1.231}$$

(Lebedev [1]);

$$f(x) = \frac{1}{\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} tK_t(x)dt \int_0^\infty I_t(y) f(y) \frac{dy}{y}, \tag{1.232}$$

(Lebedev [2]);

$$f(x) = \int_0^\infty t \tanh \pi t P_{it-1/2}(x)dt \int_1^\infty P_{it-1/2}(y) f(y)dy, \tag{1.233}$$

(Mehler [1], Fock [1]);

$$f(x) = \frac{1}{(\pi x)^2} \int_0^\infty t \sinh 2\pi t \Gamma(1/2 - \mu - it)\Gamma(1/2 - \mu + it) W_{\mu,it}(x)dt$$

$$\times \int_0^\infty W_{\mu,it}(y) f(y)dy, \tag{1.234}$$

(Wimp [1]);

$$f(x) = -\frac{4}{\pi^2} \frac{d}{dx} \int_0^\infty t \sinh \pi t K_{it}^2(x)dt$$

$$\times \int_0^\infty [I_{it}(y) + I_{-it}(y)] K_{it}(y) f(y)dy, \tag{1.235}$$

(Lebedev [8]);

$$f(x) = \frac{2i}{\pi} \int_0^\infty \frac{d}{dx} I_{it}^2(x) t\,dt$$

$$\times \int_0^\infty K_{it}^2(y) f(y)dy, \tag{1.236}$$

(Lebedev [8]);

$$f(x) = \frac{1}{\pi^2} \int_0^\infty t \sinh 2\pi t$$

$$\times G\, {}^{q-m,\, p-n+2}_{p+2,\, q} \left(x \left| \begin{matrix} \mu + it, \ \mu - it, \ -(\alpha_p^{n+1}), \ -(\alpha_n) \\ -(\beta_q^{m+1}), \ -(\beta_m) \end{matrix} \right. \right) dt$$

$$\times \int_0^\infty G\, {}^{m,\, n+2}_{p+2,\, q} \left(y \left| \begin{matrix} 1 - \mu + it, \ 1 - \mu - it, \ (\alpha_p) \\ (\beta_q) \end{matrix} \right. \right) f(y)dy, \tag{1.237}$$

(Wimp [1], Yakubovich [2]).

Thus these representations as it is easily seen generate the index transforms that we shall consider in detail in next chapters. Let us note here briefly that the results related to Mellin's transform (1.203), its convolution (1.217) and Slater's Theorem 1.6 are very useful to obtain various representations of hypergeometric functions as the kernels of the index transforms as well as to construct new kernels and index transforms. For instance, along with integrals (1.98)-(1.99), (1.105)-(1.106) for the Macdonald function we need to consider the following one

$$K_{i\tau}(x) = \frac{1}{2}\left(\frac{x}{2}\right)^{-i\tau} \int_0^\infty e^{-\frac{x^2}{4t}-t} t^{i\tau-1} dt, \; x > 0. \tag{1.238}$$

Appealing to Marichev [1] write this result through the hypergeometric function ${}_0F_1(a;x)$ using Slater's Theorem 1.6, because the integral (1.238) can be easily reduced to the Mellin convolution (1.217). Thus we obtain the expression of type

$$K_{i\tau}(x) = \Re_{i\tau}\left[\left(\frac{x}{2}\right)^{i\tau}\Gamma(-i\tau)_0F_1\left(1+i\tau;\;\frac{x^2}{4}\right)\right], \; x > 0. \tag{1.239}$$

This simple example enables us to deduce different representations for the index transforms kernels like (1.239) and to recognize many index integrals reducing theirs to the known index transform expansions. We shall use it extensively below in our further considerations.

Chapter 2

The Kontorovich-Lebedev Transform

In previous Chapter 1 we already gave some auxiliary results for our further considerations of the index transforms and slightly touched the index expansions of arbitrary functions. Here we start to study index transforms from famous *Kontorovich-Lebedev's transform* (the K-L transform) introduced by Kontorovich and Lebedev [1] and developed by Lebedev later in Lebedev [1]-[3], [7]. As it was shown by the author (see Yakubovich [1], [3]-[4], Yakubovich and Luchko [2]), the Kontorovich-Lebedev transform generates the respective class of so-called index transforms of *the Kontorovich-Lebedev type* and consequently, it is one of the basic integral transforms by index of the Macdonald function (1.98).

This chapter deals with modern results on this matter recently obtained by the author. We shall attract our attention to L_p-theory of the Kontorovich-Lebedev transform. Certain results in different functional spaces the reader can find in Yakubovich and Luchko [2] and in extensive bibliographical references of the present book. In particular, we note some other papers devoted to the Kontorovich-Lebedev transform and applications as Ben-Menahem [1], Buggle [1], Cessenat [1], Chakrabarti [1], Forristal and Ingram [1], Glaeske [4], Gomilko [1]-[3], Isaeva [1], Jones [1], Lebedev [5], [9], Lebedev and Kontorovich [1], Lisena [1], Lowndes [1]-[3], Naylor [1]-[2], Negrin [1]-[4], Orlyuk [1], Pandey [1], Pathak and Pandey [1], Pestun [1], [2], Vu Kim Tuan and Yakubovich [1]-[2], Vu Kim Tuan et al. [1], Wong [2], Yakubovich [6], Yakubovich and Vu Kim Tuan [1], Yakubovich and Fisher [1], Yakubovich et al. [1].

2.1 Definition, inversion in $L_{\nu,p}$

In this section we introduce as usually the Kontorovich-Lebedev transform refer-ring to the expansion (1.231) as

$$K_{i\tau}[f] = \int_0^\infty K_{i\tau}(y)f(y)dy, \tag{2.1}$$

where the index $\tau \geq 0$, $K_{i\tau}(y)$ is the Macdonald function (1.98), and we mean to take an arbitrary function $f(x)$ from the weighted space $L_{\nu,p}(\mathbf{R}_+)$ (1.19) with $\nu \in \mathbf{R}$ and $p \geq 1$. First of all observe from the integral representation (2.1) and definition of the Macdonald function (1.91) that the Kontorovich-Lebedev transform of the function f is even function of real variable τ and without loss of generality we can consider only nonnegative variable τ. From the asymptotic behavior of the Macdonald function given by formulae (1.96)-(1.97) and the Hölder inequality (1.21) for weighted spaces we immediately obtain that integral (2.1) is absolutely convergent for any function $f(x) \in L_{\nu,p}(\mathbf{R}_+)$ with $\nu < 1$. Namely, we have the following lemma.

Lemma 2.1. *Let $f(x)$ be from the space $L_{\nu,p}(\mathbf{R}_+)$ with $\nu < 1$. Then the following uniform estimate by $\tau \geq 0$ for the Kontorovich-Lebedev transform* (2.1) *holds*

$$|K_{i\tau}[f]| \leq C||f||_{L_{\nu,p}(\mathbf{R}_+)}, \tag{2.2}$$

where C is an absolute positive constant,

$$C = \left(\int_0^\infty K_0^q(y)y^{(1-\nu)q-1}dy\right)^{1/q}, \ q = p/(p-1) \tag{2.3}$$

and $K_0(x)$ is the Macdonald function of zero index.

Proof. To establish this estimate one can appeal to inequality (1.147) and invoking with the Hölder inequality (1.21) we have

$$|K_{i\tau}[f]| \leq \int_0^\infty K_0(y)|f(y)|dy$$

$$\leq \left(\int_0^\infty K_0^q(y)y^{(1-\nu)q-1}dy\right)^{1/q}\left(\int_0^\infty |f(y)|^p y^{\nu p-1}dy\right)^{1/p}$$

$$= C||f||_{L_{\nu,p}(\mathbf{R}_+)}. \tag{2.4}$$

Indeed, according to asymptotic formulae (1.96)-(1.97) the integral (2.3) is obviously convergent when $\nu < 1$. This completes the proof of Lemma 2.1. •

Lemma 2.1 shows that the Kontorovich-Lebedev transform of $L_{\nu,p}$-functions is at least continuous function on $\tau \in \mathbf{R}_+$ in view of uniform convergence of the integral (2.1). Moreover, we can deduce its differential properties. Precisely, performing the differentiation by τ of arbitrary order $k = 0, 1, \ldots$ under the integral sign in formula (1.98) by Lebesgue Theorem 1.2 we arrive to the following formula

$$\frac{\partial^k}{\partial \tau^k}K_{i\tau}(x) = \frac{1}{2}\int_{-\infty}^\infty e^{-x\cosh u}e^{i\tau u}(iu)^k du \tag{2.5}$$

and evidently

$$\left|\frac{\partial^k}{\partial\tau^k}K_{i\tau}(x)\right| \leq \int_0^\infty e^{-x\cosh u}u^k du. \tag{2.6}$$

Lemma 2.2. *Under conditions of Lemma 2.1 the K-L transform is an infinitely differentiable function on the nonnegative real axis and for any $k = 0,\ 1, \dots$ we have the uniform estimate of type*

$$\left|\frac{d^k}{d\tau^k}K_{i\tau}[f]\right| \leq B_k\|f\|_{L_{\nu,p}(\mathbf{R}_+)}, \tag{2.7}$$

where

$$B_k = q^{\nu-1}\Gamma^{1/q}(q(1-\nu))\int_0^\infty \frac{u^k}{\cosh^{1-\nu}u}du < +\infty,\ k = 0,\ 1, \dots . \tag{2.8}$$

Proof. As in the previous Lemma 2.1 making use the Hölder inequality (1.21) for weighted spaces we obtain

$$\left|\frac{d^k}{d\tau^k}K_{i\tau}[f]\right| \leq \left(\int_0^\infty \left|\frac{\partial^k}{\partial\tau^k}K_{i\tau}(y)\right|^q y^{(1-\nu)q-1}dy\right)^{1/q} \|f\|_{L_{\nu,p}(\mathbf{R}_+)}. \tag{2.9}$$

Invoking with the generalized Minkowski inequality (1.10) and using estimate (2.6) we continue

$$\left(\int_0^\infty \left|\frac{\partial^k}{\partial\tau^k}K_{i\tau}(y)\right|^q y^{(1-\nu)q-1}dy\right)^{1/q}$$
$$\leq \int_0^\infty u^k \left(\int_0^\infty e^{-qy\cosh u}y^{(1-\nu)q-1}dy\right)^{1/q} du$$
$$= q^{\nu-1}\Gamma^{1/q}(q(1-\nu))\int_0^\infty \frac{u^k}{\cosh^{1-\nu}u}du = B_k < +\infty,\ \nu < 1. \tag{2.10}$$

Lemma 2.2 is proved. •

From the above properties for the K-L transform it follows that we can discuss its belonging to $L_r(\mathbf{R}_+)$-space for some $1 \leq r \leq \infty$ investigating only its behavior at infinity. The answer can be obtained applying more precise estimate (1.100).

Lemma 2.3. *The operator of the K-L transform* (2.1) *is a bounded mapping from any space $L_{\nu,p}(\mathbf{R}_+)$, $\nu < 1$, $p \geq 1$ into the space $L_r(\mathbf{R}_+)$, where $r \geq 1$ and the parameters p, r have no dependence.*

Proof. Indeed, treating like Lemma 2.1 with using estimate (1.100) we establish the following inequality

$$|K_{i\tau}[f]| \leq e^{-\delta\tau}\int_0^\infty K_0(y\cos\delta)|f(y)|dy$$

$$\leq e^{-\delta\tau} \left(\int_0^\infty K_0^q(y\cos\delta) y^{(1-\nu)q-1} dy \right)^{1/q} \left(\int_0^\infty |f(y)|^p y^{\nu p-1} dy \right)^{1/p}$$

$$= C_\delta e^{-\delta\tau} ||f||_{L_{\nu,p}(\mathbf{R}_+)}, \tag{2.11}$$

where the constant $C_\delta > 0$ and it depends from $\delta \in [0, \pi/2)$. Thus from (2.11) it is obviously to see that the norm like (1.1) for the K-L transform (2.1) in space $L_r(\mathbf{R}_+)$, $r \geq 1$ is finite putting in (2.11) any fixed number $\delta \in (0, \pi/2)$. Moreover, we established the fact that the K-L transform belongs to weighted space $L_r(\mathbf{R}_+; \rho)$, if the weighted function $\rho(\tau)$ satisfies the condition

$$\int_0^\infty \rho(\tau) e^{-\delta r \tau} d\tau < \infty. \tag{2.12}$$

So we led to the desired result. Lemma 2.3 is proved. •

These lemmas show that the K-L transform (2.1) of an arbitrary $L_{\nu,p}$- function $f(x)$ possesses as well as the smoothness and L_r-properties and furthermore, it is evident fact that the range of the K-L transform (we denote it as $KL(L_{\nu,p})$), precisely

$$KL(L_{\nu,p}) = \{g : g(\tau) = K_{i\tau}[f],\ f \in L_{\nu,p}(\mathbf{R}_+)\},\ \nu < 1,\ p \geq 1 \tag{2.13}$$

does not coincide with the space $L_r(\mathbf{R}_+)$. Indeed, we know that the K-L transform belongs to the weighted space $L_r(\mathbf{R}_+; \rho)$ too with condition (2.12). But choosing different weights one can easily verify that there exists some function, which belongs to $L_r(\mathbf{R}_+)$, but does not belong to the space $L_r(\mathbf{R}_+; \rho)$ and vice versa. Thus it is necessary to describe the range of the K-L transform (2.1).

For this purpose we shall use the inverse operator formally following to expansion (1.230). It was introduced in slightly different form in Yakubovich and Luchko [2], Chapter 6. Namely, let us consider the operator of type

$$(I_\varepsilon g)(x) = \frac{2}{\pi^2 x^{1-\varepsilon}} \int_0^\infty \tau \sinh((\pi - \varepsilon)\tau) K_{i\tau}(x) g(\tau) d\tau, \tag{2.14}$$

where $\varepsilon \in (0, \pi)$.

Theorem 2.1. *On functions $g(\tau) = K_{i\tau}[f]$ which are represented by the Kontorovich-Lebedev transform (2.1) with the density $f(y) \in L_{\nu,p}(\mathbf{R}_+)$, $\nu < 1$, $1 \leq p \leq \infty$, the operator (2.14) has the following form*

$$(I_\varepsilon g)(x) = \frac{\sin\varepsilon}{\pi x^{-\varepsilon}} \int_0^\infty \frac{K_1((x^2 + y^2 - 2xy\cos\varepsilon)^{1/2})}{(x^2 + y^2 - 2xy\cos\varepsilon)^{1/2}} y f(y) dy,\ x > 0, \tag{2.15}$$

where $K_1(z)$ is the Macdonald function (1.91) of the order 1.

Proof. Substituting the value of $g(\tau)$ as the K-L transform (2.1) in formula (2.14) and appealing to inequality (1.100) we have the estimate as

$$|(I_\varepsilon g)(x)| \leq \frac{2}{\pi^2 x^{1-\varepsilon}} K_0(x\cos\delta_1) \int_0^\infty \tau \sinh((\pi - \varepsilon)\tau) e^{-(\delta_1+\delta_2)\tau} d\tau$$

$$\times \int_0^\infty K_0(y\cos\delta_2)|f(y)|dy \tag{2.16}$$

provided that we choose $\delta_1 + \delta_2 + \varepsilon > \pi$. Obviously two integrals in (2.16) are convergent (the second one is provided by Lemma 2.3). Hence we can apply Fubini's Theorem 1.1 and interchange the order of integration in obtained iterated integral. As a result we use formula 2.16.51.8 from Prudnikov et al. [2], namely

$$\int_0^\infty \tau \sinh((\pi-\varepsilon)\tau)K_{i\tau}(x)K_{i\tau}(y)d\tau$$

$$= \frac{\pi xy\sin\varepsilon}{2}\frac{K_1((x^2+y^2-2xy\cos\varepsilon)^{1/2})}{(x^2+y^2-2xy\cos\varepsilon)^{1/2}} \tag{2.17}$$

which gives us representation (2.15). This completes the proof of Theorem 2.1. •

The inversion formula of the K-L transform (2.1) at the space $L_{\nu,p}(\mathbf{R}_+)$ is established by our next theorem.

Theorem 2.2. *Let* $g(\tau) = K_{i\tau}[f]$, $f(y) \in L_{\nu,p}(\mathbf{R}_+)$, $0 < \nu < 1$, $1 \leq p \leq \infty$. *Then*

$$f(x) = (Ig)(x), \tag{2.18}$$

where $(Ig)(x)$ *is understood as*

$$(Ig)(x) = \text{l.i.m.}_{\varepsilon\to 0+}(I_\varepsilon g)(x),\ x > 0 \tag{2.19}$$

and the limit in (2.19) *is meant in terms of the norm in* $L_{\nu,p}(\mathbf{R}_+)$ *by formula* (1.19). *Moreover, the limit in* (2.19) *also exists almost everywhere on* $\mathbf{R}_+$.

Proof. Considering the integral (2.15) by replacement of variable $y = x(\cos\varepsilon + t\sin\varepsilon)$, we arrive to the following equality

$$(I_\varepsilon g)(x) = \frac{1}{\pi}\int_{-\infty}^\infty \frac{R(x,t,\varepsilon)}{t^2+1} f(x(\cos\varepsilon + t\sin\varepsilon))(\cos\varepsilon + t\sin\varepsilon)dt, \tag{2.20}$$

where we denote by $R(x,t,\varepsilon)$ the function of type

$$R(x,t,\varepsilon) = \begin{cases} x^{\varepsilon+1}\sin\varepsilon(t^2+1)^{1/2}K_1(x\sin\varepsilon(t^2+1)^{1/2}),\ t \geq -\cot\varepsilon, \\ 0,\ t < -\cot\varepsilon. \end{cases} \tag{2.21}$$

From the asymptotic behavior (1.96)-(1.97) of the Macdonald function $K_1(z)$ we obtain that for any $t \in \mathbf{R}$, $x \in \mathbf{R}_+$, and $\varepsilon \in (0,\pi)$, $R(x,t,\varepsilon) < C$ uniformly as the function of three variables. In addition, we observe the limit equality

$$\lim_{\varepsilon\to 0+} R(x,t,\varepsilon) = 1.$$

Further, we use the approximation properties of the Poisson kernel (1.14) and Theorem 1.4 to estimate the $L_{\nu,p}$-norm $0 < \nu < 1$, $1 \leq p \leq \infty$ (1.19) of difference $(I_\varepsilon g) - f$ applying the generalized Minkowski inequality (1.10). Namely, find that

$$||\,(I_\varepsilon g) - f||_{L_{\nu,p}(\mathbf{R}_+)} \leq \frac{1}{\pi}\int_{-\infty}^\infty \frac{1}{t^2+1}||f(x(\cos\varepsilon + t\sin\varepsilon)(\cos\varepsilon$$

$$+t\sin\varepsilon)R(x,t,\varepsilon)-f(x)||_{L_{\nu,p}(\mathbf{R}_+)}dt \to 0,\ \varepsilon\to 0+. \tag{2.22}$$

Moreover, we have the estimate from (2.20) as

$$||\,(I_\varepsilon g)\,||_{L_{\nu,p}(\mathbf{R}_+)} < \frac{C}{\pi}\int_{-\cot\varepsilon}^{\infty}\frac{1}{t^2+1}||f(x(\cos\varepsilon+t\sin\varepsilon))$$

$$\times(\cos\varepsilon+t\sin\varepsilon)||_{L_{\nu,p}(\mathbf{R}_+)}dt$$

$$< C||f||_{L_{\nu,p}(\mathbf{R}_+)}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(1+\mid t\mid)^{1-\nu}}{t^2+1}dt = C_1||f||_{L_{\nu,p}(\mathbf{R}_+)},\ 0<\nu<1, \tag{2.23}$$

where C_1 is a positive absolute constant, because the integral by t is convergent under condition on parameter ν. Thus from Lebesgue's Theorem 1.2 and the continuity of the $L_{\nu,p}$-norm we proved the equality (2.19). The existence of the limit almost everywhere on $\mathbf{R}_+$ follows from the radial property of the Poisson kernel (1.14) $P(t)=P(|t|)\in L_1(\mathbf{R}_+)$ and Theorem 1.3, which can be formulated without difficulties also for $L_{\nu,p}$-spaces . Theorem 2.2 is proved. •

Theorem 2.2 yields the inequality

$$||\,(I_\varepsilon g)\,||_{L_{\nu,p}(\mathbf{R}_+)} \le C||\,(Ig)\,||_{L_{\nu,p}(\mathbf{R}_+)}, \tag{2.24}$$

where

$$g\in KL(L_{\nu,p}),\ 0<\nu<1,\ 1\le p\le\infty.$$

It follows from Theorem 2.2 that $K_{i\tau}[f]\equiv 0, f(y)\in L_{\nu,p}(\mathbf{R}_+), 0<\nu<1,\ 1\le p\le\infty$, iff $f(y)=0$ almost everywhere on $\mathbf{R}_+$. So, in the space $KL(L_{\nu,p})$ one can introduce a norm by the equality

$$||g||_{KL(L_{\nu,p})} = ||f||_{L_{\nu,p}},\ g=K_{i\tau}[f]. \tag{2.25}$$

As it is evident, the space $KL(L_{\nu,p})$ is a Banach one with norm (2.25) and as an isometric to $L_{\nu,p}$.

The next theorem of this section gives the characterization of the space $KL(L_{\nu,p})$ in terms of operators (2.14).

Theorem 2.3. *The necessary and sufficient conditions for $g(\tau)\in KL(L_{\nu,p}), 0<\nu<1,\ 1\le p\le\infty$ are*

$$g(\tau)\in L_r(\mathbf{R}_+), 1\le r\le\infty, \tag{2.26}$$

$$\text{l.i.m.}_{\varepsilon\to 0+}(I_\varepsilon g)\in L_{\nu,p}(\mathbf{R}_+). \tag{2.27}$$

Proof. The necessity in this theorem is a simple fact, being a corollary of Lemma 2.3, of Theorem 2.2 and of inequality (2.24). The sufficiency part is more complicated.

Let $g(\tau)\in L_r(\mathbf{R}_+)$ and assume that condition (2.27) is valid. We are to show that there exists a function $f\in L_{\nu,p}$, such that

$$g=K_{i\tau}[f] \tag{2.28}$$

(then $g \in KL(L_{\nu,p})$). From condition (2.27) conclude that $(I_\varepsilon g) \in L_{\nu,p}$ for sufficiently small $\varepsilon \in (0, \pi)$ and one can calculate the following composition

$$K_{i\tau}[(I_\varepsilon g)] = \int_0^\infty K_{i\tau}(y)\,(I_\varepsilon g)\,(y)dy. \tag{2.29}$$

Take functions $g(\tau)$ being sufficiently good, e.g. smooth functions with compact support C_0^∞ on $\mathbf{R}_+$, the set of whose is dense in L_r. Hence we have by substituting (2.14) into equality (2.29) the possibility to change the order of integration by the Fubini Theorem 1.1. Using the value of the integral 2.16.33.2 from Prudnikov et al. [2], namely

$$\int_0^\infty y^{\varepsilon-1} K_{i\tau}(y) K_{i\beta}(y) dy$$
$$= \frac{2^{\varepsilon-3}}{\Gamma(\varepsilon)} \left| \Gamma\left(\frac{\varepsilon + i(\tau+\beta)}{2}\right) \Gamma\left(\frac{\varepsilon + i(\tau-\beta)}{2}\right) \right|^2, \tag{2.30}$$

we obtain

$$K_{i\tau}[(I_\varepsilon g)] = g_\varepsilon(\tau) = \frac{2^{\varepsilon-2}}{\pi^2 \Gamma(\varepsilon)} \int_0^\infty \beta \sinh((\pi-\varepsilon)\beta)$$
$$\times \left| \Gamma\left(\frac{\varepsilon + i(\tau+\beta)}{2}\right) \Gamma\left(\frac{\varepsilon + i(\tau-\beta)}{2}\right) \right|^2 g(\beta) d\beta. \tag{2.31}$$

In order to prove the validity of equality (2.31) for all $g \in L_r(\mathbf{R}_+)$ we may prove now the boundedness of the operator in the right-hand side of (2.31) and use after the Banach Theorem 1.5. But as it is not difficult to see from the asymptotic formula for gamma-function (1.33) the kernel of the integrand in (2.31) is equal to

$$O(e^{(\pi/2-\varepsilon)\beta - \pi|\tau-\beta|/2 - \pi\tau/2}),\ (\beta,\ \tau) \in \mathbf{R}_+ \times \mathbf{R}_+, \varepsilon \in (0,\ \pi). \tag{2.32}$$

Hence we have the following estimate

$$|\ K_{i\tau}[(I_\varepsilon g)]\ | < Ce^{-\pi\tau/2} \int_0^\infty e^{(\pi/2-\varepsilon)\beta - \pi|\beta-\tau|/2} |g(\beta)| d\beta$$
$$< Ce^{(\delta-\pi/2)\tau} \int_0^\infty e^{(\pi/2-\varepsilon-\delta)\beta} |g(\beta)| d\beta, \tag{2.33}$$

where the value of some parameter δ is taken from the interval $(\pi/2-\varepsilon,\ \pi/2)$. So from estimate (2.33) with the aid of the Hölder inequality, we establish the boundedness of the operator in the right-hand side of (2.31) in the space $L_r(\mathbf{R}_+)$, $1 \le r \le \infty$.

Now let us calculate the limit of the right-hand side of (2.31), when $\varepsilon \to 0+$ in norm of the space $L_r(\mathbf{R}_+)$. We begin by representing the function $g_\varepsilon(\tau)$ after substitution $\beta = \tau + \varepsilon t$ as follows

$$g_\varepsilon(\tau) = \frac{1}{\pi} \int_{-\infty}^\infty \frac{g(\tau+\varepsilon t)}{t^2+1} h(\tau,\ t, \varepsilon) dt, \tag{2.34}$$

where

$$h(\tau, t, \varepsilon) = H(\tau + \varepsilon t) \frac{2^{\varepsilon-2} \varepsilon (\tau+\varepsilon t)(t^2+1) \sinh((\pi-\varepsilon)(\tau+\varepsilon t))}{\pi \Gamma(\varepsilon)}$$

$$\times \left| \Gamma\left(i\tau + \frac{\varepsilon}{2}(1+it)\right) \Gamma\left(\frac{\varepsilon}{2}(1-it)\right)\right|^2 \tag{2.35}$$

and $H(x)$ as usually is the Heaviside function. From the previous discussion, we conclude that the function $h(\tau, t, \varepsilon)$ is bounded uniformly for all parameters $\tau > 0$, $t \in \mathbf{R}$, $\varepsilon \in (0, \pi)$. Moreover, from the supplement formulae (1.29) for the gamma-function the following limit relation takes place

$$\lim_{\varepsilon \to 0+} h(\tau, t, \varepsilon) = 1. \tag{2.36}$$

Hence we obtain the following estimate for norm of the function $g_\varepsilon(\tau)$ in the space $L_r(\mathbf{R}_+)$

$$||g_\varepsilon(\tau) - g(\tau)||_{L_r(\mathbf{R}_+)}$$

$$\leq \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{t^2+1} ||g(\tau + \varepsilon t) h(\tau, t, \varepsilon) - g(\tau)||_{L_r(\mathbf{R}_+)} dt \to 0, \ \varepsilon \to 0+. \tag{2.37}$$

However, on the other side, appealing to estimate (2.11) we see that operator of the K-L transform (2.1) is a bounded mapping in $L_{\nu,p}$-space, where $0 < \nu < 1$, $1 \leq p \leq \infty$, because due to inequality (2.12) the weight $\rho(\tau) = \tau^{\nu p - 1}$ satisfies this condition. Thus there exists the following limit in $L_{\nu,p}$-norm

$$\text{l.i.m.}_{\varepsilon \to 0+} K_{i\tau}[(I_\varepsilon g)] = K_{i\tau}[\text{l.i.m.}_{\varepsilon \to 0+} (I_\varepsilon g)] = K_{i\tau}[f], \tag{2.38}$$

where $f = Ig \in L_{\nu,p}$. Since the operator $K_{i\tau}[(I_\varepsilon g)]$ converges in the norm L_r too, then limit functions must coincide almost everywhere on $\mathbf{R}_+$. Thus, from equality (2.38) we obtain (2.28). Theorem 2.3 is completely proved. •

By means of Theorem 2.3 we already described the range of the Kontorovich-Lebedev transform (2.1) of the space $L_{\nu,p}(\mathbf{R}_+)$, when $0 < \nu < 1$ and $p \geq 1$. However, as is known in the theory of integral transforms the very important case is $p = 2$, when the Lebesgue space $L_{\nu,p}$ becomes the Hilbert space with respect to the inner product of two in general complex-valued functions as

$$\langle f, g \rangle = \int_0^\infty x^{2\nu - 1} f(x) \overline{g(x)} dx \tag{2.39}$$

and the norm $||f||_{\nu,2} = \sqrt{\langle f, f \rangle}$. For the K-L transform the most transparent case is $\nu = 1$ that is the limit case for the above theorems. Nevertheless, we can describe exactly the range $KL(L_{1,2})$ and be succeeded in proving of the Parseval equality like for instance (1.193) for the Fourier transform when $p = q = 2$ (it becomes equality for these values).

2.2 Note on the K-L-summability of integrals

In the previous section we investigated the K-L transform at $L_{\nu,p}$ space and described the range of this transform. We started from expansion (1.231) to define the K-L transform (2.1). As it was established earlier (see Lebedev [1], Yakubovich and Luchko [2]) the iterated integral (1.231) is meant in definite sense. Usually the inside integral by y is absolutely convergent and outside integral is understood in principal value sense as, for instance in the theory of Fourier integrals from Titchmarsh [1]. Nevertheless, one can find there the notion of the summability of integrals. In particular, for Fourier integrals we have several types of such summability, namely (Titchmarsh [1]) *Fejér's Integral*

$$f(x) = \lim_{\lambda\to\infty} \frac{1}{\pi}\int_0^\lambda \left(1-\frac{u}{\lambda}\right) du \int_{-\infty}^\infty f(t)\cos(u(x-t))dt$$

$$= \lim_{\lambda\to\infty} \frac{1}{\pi}\int_{-\infty}^\infty f(t)\frac{2\sin^2\left(\frac{\lambda}{2}(x-t)\right)}{\lambda(x-t)^2}dt, \qquad (2.39)'$$

Cauchy's Singular Integral

$$f(x) = \lim_{\varepsilon\to+0} \frac{1}{\pi}\int_{-\infty}^\infty f(t)dt \int_0^\infty e^{-\varepsilon u}\cos(u(x-t))du$$

$$= \lim_{\varepsilon\to+0} \frac{1}{\pi}\int_{-\infty}^\infty f(t)\frac{\varepsilon}{\varepsilon^2+(x-t)^2}dt, \qquad (2.40)$$

Weierstrass's Singular Integral

$$f(x) = \lim_{\varepsilon\to+0} \frac{1}{\pi}\int_{-\infty}^\infty f(t)dt \int_0^\infty e^{-\varepsilon^2 u^2}\cos(u(x-t))du$$

$$= \lim_{\varepsilon\to+0} \frac{1}{2\varepsilon\sqrt{\pi}}\int_{-\infty}^\infty f(t)\exp\left(-\frac{(x-t)^2}{4\varepsilon^2}\right)dt. \qquad (2.41)$$

These formulae are valid under different conditions and in different spaces of functions. By Theorem 1.3 we demonstrated also some general notion of the summability through the notion of the averaging.

Thus we have the right to consider the introduced operator (2.14) and its representation (2.15) by Theorem 2.1 as *Kontorovich-Lebedev's Singular Integral*, namely by meaning the limit in different senses we have the equality

$$f(x) = \lim_{\varepsilon\to+0} \frac{2}{\pi^2 x^{1-\varepsilon}}\int_0^\infty \tau\sinh((\pi-\varepsilon)\tau)K_{i\tau}(x)\int_0^\infty K_{i\tau}(y)f(y)dyd\tau$$

$$= \lim_{\varepsilon\to+0} \frac{\sin\varepsilon}{\pi x^{-\varepsilon}}\int_0^\infty \frac{K_1((x^2+y^2-2xy\cos\varepsilon)^{1/2})}{(x^2+y^2-2xy\cos\varepsilon)^{1/2}}yf(y)dy, \; x>0. \qquad (2.42)$$

2.3 The space $L_{1,2}(\mathbf{R}_+)$. Parseval's relation. Plancherel's theorem

In this section we consider the K-L transform (2.1) at the weighted Hilbert space $L_{1,2}(\mathbf{R}_+)$. As we seek from the inversion Theorem 2.2 this case of the Lebesgue space is the limit case when $\nu = 1$ and integral (2.1) does not exist as absolutely convergent improper integral. For example, let us take the function $f(x) = e^{-x}/x,\ x > 0$. Then as we can see from the definition of the norm $L_{1,2}(\mathbf{R}_+)$, by formula (1.19) the function e^{-x}/x belongs to this space although the integral (2.1) is divergent in usual sense (see the asymptotic of the Macdonald function (1.97) at the neighborhood of the point $x = 0$). Nevertheless, we shall show that the Kontorovich-Lebedev transform (2.1) can be defined as follows

$$K_{i\tau}[f] = \text{l.i.m.}_{N\to\infty} \int_{1/N}^{N} K_{i\tau}(y) f(y) dy, \tag{2.43}$$

where $f(y) \in L_{1,2}(\mathbf{R}_+)$ and the limit in (2.43) is meant by some weighted Hilbert space too, which we shall define later. Obviously, if $f(x) \in L_{1,2}(\mathbf{R}_+)$, then $f(x) \in L_{1,2}([1/N, N])$ for any number $N > 0$. Moreover, we have the estimate

$$\int_{1/N}^{N} y^{2\nu-1} |f(y)|^2 dy < C \int_{1/N}^{N} y |f(y)|^2 dy = C ||f||^2_{L_{1,2}([1/N,N])}, \tag{2.44}$$

where C is an absolute positive constant and it gives that $f(x) \in L_{\nu,2}([1/N, N])$ for any $0 < \nu < 1$. Therefore, according to Lemma 2.1 integral (2.43) is absolutely convergent and the Kontorovich-Lebedev transform (2.1) of the function $f_N = f(x),\ x \in [1/N, N],\ f(x) = 0,\ 0 < x < 1/N$ exists.

As usually let us define the inner product of two complex-valued functions $f(x), g(x)$ of the Hilbert space $L_{1,2}(\mathbf{R}_+)$ by formula

$$\langle f, g\rangle = \int_0^\infty y f(y) \overline{g(y)} dy. \tag{2.45}$$

We shall prove that the range of the K-L transform (2.43) coincides with the weighted Hilbert space $L_2\left(\mathbf{R}_+; \frac{2}{\pi^2}\tau \sinh(\pi\tau)\right)$ normed by

$$||h||_{L_2\left(\mathbf{R}_+; \frac{2}{\pi^2}\tau \sinh(\pi\tau)\right)} = \frac{\sqrt{2}}{\pi} \left(\int_0^\infty \tau \sinh(\pi\tau) |h(\tau)|^2 d\tau\right)^{1/2} \tag{2.46}$$

and the limit in (2.43) is understood by means of the convergence by norm (2.46). Considering the respective inner product $(K_{i\tau}[f], K_{i\tau}[g])$ of the K-L images in the space (2.46) we have formally the chain of equalities

$$(K_{i\tau}[f], K_{i\tau}[g]) = \frac{2}{\pi^2} \int_0^\infty \tau \sinh(\pi\tau) K_{i\tau}[f] \overline{K_{i\tau}[g]} d\tau$$

$$= \frac{2}{\pi^2}\int_0^\infty \tau\sinh(\pi\tau)K_{i\tau}[f]\overline{\int_0^\infty K_{i\tau}(y)g(y)dyd\tau}$$

$$= \int_0^\infty \overline{g(y)}dy\frac{2}{\pi^2}\int_0^\infty \tau\sinh(\pi\tau)K_{i\tau}(y)K_{i\tau}[f]d\tau$$

$$= \int_0^\infty yf(y)\overline{g(y)}dy = \langle f, g\rangle, \tag{2.47}$$

where we use the Kontorovich-Lebedev singular integral (2.42) passing to the limit under the sign of the integral or appealing to expansion (1.231). The corresponding conditions of the validity of equalities (2.47) are given for example, by the following

Lemma 2.4. *If* $g(x) \in L_1(\mathbf{R}_+; K_0(x\cos\delta))$ *and* $K_{i\tau}[f] \in L_1(\mathbf{R}_+; \tau\exp((\pi - \delta)\tau))$, $\delta \in [0, \pi/2)$, *then the equality between inner products takes place, namely*

$$(K_{i\tau}[f], K_{i\tau}[g]) = \langle f, g\rangle. \tag{2.48}$$

Remark 2.1. Equality (2.48) is called as usually *the Parseval equality* for the Kontorovich-Lebedev transform (2.1).

Proof. The proof of this lemma can be easily obtained by Fubini's theorem under the above conditions, which provide the absolute convergence of the iterated integrals at the chain of equalities (2.47) and can be written invoking with inequality (1.100). This completes the proof of Lemma 2.4. •

Remark 2.2. Letting $f = g$ at the previous lemma we obtain that $f \in L_{1,2}(\mathbf{R}_+)$ and $h(\tau) = K_{i\tau}[f] \in L_2\left(\mathbf{R}_+; \frac{2}{\pi^2}\tau\sinh(\pi\tau)\right)$ with equality for the corresponding norms as

$$||f||_{L_{1,2}(\mathbf{R}_+)} = ||h||_{L_2\left(\mathbf{R}_+; \frac{2}{\pi^2}\tau\sinh(\pi\tau)\right)}. \tag{2.49}$$

Let us turn now to the space $C^{(2)}(\mathbf{R}_+)$ of the smooth functions of the order two with compact support on $\mathbf{R}_+$. As is known, for instance, in Titchmarsh [1] the space $C^{(2)}(\mathbf{R}_+)$ is dense everywhere in $L_{1,2}(\mathbf{R}_+)$.

Lemma 2.5. *If the function* $f(x) \in C^{(2)}(\mathbf{R}_+)$ *and its support is compact one, then the Kontorovich-Lebedev transform* (2.1) *belongs to the space* $L_1(\mathbf{R}_+; \sqrt{\tau}\exp(\pi\tau/2))$.

Proof. It is clear that $f(x)$ satisfies to the conditions of Lemma 2.1 and consequently, $K_{i\tau}[f]$ is continuous function of variable $\tau \in \mathbf{R}_+$. The compact support of the function $f(x)$ allows us to use Theorem 1.7 of asymptotic behavior of the Macdonald function $K_{i\tau}(x)$ by index $\tau \to +\infty$. Indeed, making use formula (1.148) we obtain

$$K_{i\tau}[f] = \frac{e^{-\pi\tau/2}}{\sqrt{\tau}}O\left(\int_0^\infty e^{i\tau\log y}f(y)dy\right)$$

$$= \frac{e^{-\pi\tau/2}}{\sqrt{\tau}} O\left(\int_{-\infty}^{\infty} e^{i\tau t} f(e^t) e^t dt\right), \tau \to +\infty. \tag{2.50}$$

Denoting by $f_1(t) = f(e^t)e^t$ conclude that $f_1(t)$ has the compact support too on $\mathbf{R}$, if the function $f(x)$ possesses by this property at the half axis $\mathbf{R}_+$ and $f_1(x) \in C^{(2)}(\mathbf{R})$. Hence we appeal to the known result from the Fourier transform theory (see, for example Knyazev [1]) of the smooth functions from the space $C^{(2)}(\mathbf{R})$ with compact support that gives the final estimate for the K-L transform as

$$|K_{i\tau}[f]| < C\frac{e^{-\pi\tau/2}}{\tau^{5/2}}, \tau > 0, \tag{2.51}$$

where $C > 0$ is an absolute constant. Thus we obtain the proof of the Lemma 2.5. •

Corollary 2.1. *For functions $f(x)$ from the space $C^{(2)}(\mathbf{R}_+)$ the Parseval equality* (2.48) *is true.*

Proof. Actually, in this case for a function $f(x)$ the inversion Theorem 2.2 takes place and according to Lemma 2.5 and asymptotic formula (1.148) by index of the Macdonald function we perform to pass to the limit under the sign of integral in formula (2.14) due the Lebesgue theorem which gives us the representation of type

$$xf(x) = \frac{2}{\pi^2}\int_0^{\infty} \tau \sinh(\pi\tau) K_{i\tau}(x) K_{i\tau}[f] d\tau, \tag{2.52}$$

where the last integral is absolutely convergent. Hence at the chain of equalities (2.47) the changing of the order of integration is possible due to Fubini's theorem and we obtain the Parseval equality (2.48). Corollary 2.1 is proved. •

Let now $f(x)$ be an arbitrary function from the space $L_{1,2}(\mathbf{R}_+)$. Choose some sequence of functions from the space $C^{(2)}(\mathbf{R}_+)$ with the compact support that is convergent to the given function f by the norm of the space $L_{1,2}(\mathbf{R}_+)$. Let us denote through f_n the common term of this sequence and through the symbol I_n the least segment which contains the support of the function f_n. Since the operator of the K-L transform is linear one, then from the previous corollary we obtain the equality

$$\int_0^{\infty} x|f_n(x) - f_m(x)|^2 dx = \frac{2}{\pi^2}\int_0^{\infty} \tau \sinh(\pi\tau)|K_{i\tau}[f_n] - K_{i\tau}[f_m]|^2 d\tau. \tag{2.53}$$

In fact, as the left-hand side of equality (2.53) tends to zero by $m, n \to \infty$, therefore the sequence $\{K_{i\tau}[f_n]\}$ is the Cauchy sequence. The completeness of the respective Hilbert space $L_2\left(\mathbf{R}_+; \frac{2}{\pi^2}\tau \sinh(\pi\tau)\right)$ means the existence of the function $h(\tau) \equiv K_{i\tau}[f] \in L_2\left(\mathbf{R}_+; \frac{2}{\pi^2}\tau \sinh(\pi\tau)\right)$ such that $K_{i\tau}[f_n] \to h(\tau)$ by the norm of this space. Since

$$K_{i\tau}[f_n] = \int_{I_n} K_{i\tau}(y) f_n(y) dy, \tag{2.54}$$

then integrating the function $K_{it}[f_n]$ by segment $[0, \tau]$ we obtain

$$\int_0^{\tau} K_{it}[f_n] dt = \int_{I_n} f_n(y) dy \int_0^{\tau} K_{it}(y) dt$$

$$= \int_0^\infty K(\tau, y) f_n(y) dy, \tag{2.55}$$

where

$$K(\tau, y) = \int_0^\tau K_{it}(y) dt. \tag{2.56}$$

Let us consider the left-hand side of equality (2.55). As $K_{it}[f_n]$ belongs to $L_2\left(\mathbf{R}_+; \frac{2}{\pi^2} t \sinh(\pi t)\right)$, consequently $K_{it}[f_n] \in L_2([0;\tau])$. Since $K_{it}[f_n] \to K_{it}[f]$ by the norm of the space $L_2\left(\mathbf{R}_+; \frac{2}{\pi^2} t \sinh(\pi t)\right)$ and

$$\int_0^\tau |K_{it}[f_n] - K_{it}[f]|^2 dt < C||K_{it}[f_n] - K_{it}[f]||^2_{L_2\left(\mathbf{R}_+; \frac{2}{\pi^2} t \sinh(\pi t)\right)}, \tag{2.57}$$

then $K_{it}[f_n] \to K_{it}[f]$ by the norm $L_2([0;\tau])$. Hence by the Cauchy-Schwarz-Bunyakovskii inequality that is a particular case of the Hölder inequality (1.21) for the Hilbert spaces we have

$$\left| \int_0^\tau (K_{it}[f_n] - K_{it}[f]) dt \right| \le \int_0^\tau |K_{it}[f_n] - K_{it}[f]| dt$$

$$\le \sqrt{\tau} ||K_{it}[f_n] - K_{it}[f]||_{L_2([0;\tau])}. \tag{2.58}$$

Therefore,

$$\lim_{n\to\infty} \int_0^\tau K_{it}[f_n] dt = \int_0^\tau K_{it}[f] dt. \tag{2.59}$$

Similarly we establish the limit at the right-hand side of (2.55). Indeed, the function $f_n(x) \in L_{1,2}(\mathbf{R}_+)$ and invoking with inequality (1.21) we obtain the estimate

$$\int_0^\infty |K(\tau, y) f_n(y)| dy \le \left(\int_0^\infty \frac{|K(\tau, y)|^2}{y} dy \right)^{1/2} ||f_n||_{L_{1,2}(\mathbf{R}_+)}. \tag{2.60}$$

Thus the problem arises to show that for each $\tau > 0$ the function $K(\tau, y) \in L_{0,2}(\mathbf{R}_+)$. In fact, as is obvious from (2.56) and the simple inequality

$$|K(\tau, y)| \le \tau K_0(y) \tag{2.61}$$

the function $K(\tau, y)$ belongs to the space $L_{0,2}([a, \infty))$, where $a > 0$ is some fixed number. To prove that $K(\tau, y) \in L_{0,2}((0, a])$ we use integral (1.106). Indeed, invoking with the mean value theorem and substituting (1.106) into (2.56) we obtain

$$K(\tau, y) = C_\tau \int_0^\tau \sinh\left(\frac{\pi t}{2}\right) K_{it}(y) dt$$

$$= C_\tau \int_0^\tau dt \int_0^\infty \sin(y \sinh u) \sin(tu) du, \tag{2.62}$$

where C_τ is some constant, which depends from τ and arises by the mean value theorem. The Abel's test of uniform convergence of the integrals enables us to perform the integration by t and it gives the representation

$$K(\tau, y) = C_\tau \int_0^\infty \sin(y \sinh u) \frac{1 - \cos(\tau u)}{u} du$$

$$= C_\tau \int_0^\infty \sin(yv) \frac{1-\cos\left(\tau \log\left(v+(v^2+1)^{1/2}\right)\right)}{\log\left(v+(v^2+1)^{1/2}\right)\sqrt{v^2+1}} dv, \tag{2.63}$$

where we let the replacement $v = \sinh u$. Hence decomposing the integral (2.63) as two by $v \in (0, y]$ and $v \in [y, \infty)$ we estimate each one, namely

$$\left| \int_0^y \sin(yv) \frac{1-\cos\left(\tau \log\left(v+(v^2+1)^{1/2}\right)\right)}{\log\left(v+(v^2+1)^{1/2}\right)\sqrt{v^2+1}} dv \right|$$

$$< Cy \int_0^a \frac{v}{\log\left(v+(v^2+1)^{1/2}\right)\sqrt{v^2+1}} dv = O(y), \ 0 < y < a, \tag{2.64}$$

$$\left| \int_y^\infty \sin(yv) \frac{1-\cos\left(\tau \log\left(v+(v^2+1)^{1/2}\right)\right)}{\log\left(v+(v^2+1)^{1/2}\right)\sqrt{v^2+1}} dv \right|$$

$$= O\left(\frac{1}{\log y} \int_1^M \frac{\sin t}{t} dt\right) = O\left(\frac{1}{\log y}\right), \ 0 < y < a < 1. \tag{2.65}$$

Note here, that we treated integral (2.65) by using the second mean value theorem. Thus we can choose $0 < a < 1$ and conclude that $K(\tau, y) \in L_{0,2}((0, a])$ from the above estimates. So finally $K(\tau, y) \in L_{0,2}(\mathbf{R}_+)$. From the relation $f_n \to f$ by the norm of $L_{1,2}(\mathbf{R}_+)$ and the Cauchy-Schwarz-Bunyakovskii inequality we have that

$$\lim_{n\to\infty} \int_0^\infty K(\tau, y) f_n(y) dy = \int_0^\infty K(\tau, y) f(y) dy, \tag{2.66}$$

and after passage to the limit in the equality (2.55) it gives us

$$\int_0^\tau K_{it}[f] dt = \int_0^\infty K(\tau, y) f(y) dy. \tag{2.67}$$

Since $K_{it}[f] \in L_2\left(\mathbf{R}_+; \frac{2}{\pi^2} t \sinh(\pi t)\right)$, then $K_{it}[f] \in L_2((0, N])$ and therefore $K_{it}[f] \in L_1((0, N])$. Consequently, one can differentiate through in equality (2.67) by the upper limit. As is known, the respective derivative equals almost everywhere on $\mathbf{R}_+$ the integrand. So finally for almost all $\tau > 0$ we obtain the formula

$$K_{i\tau}[f] = \frac{d}{d\tau} \int_0^\infty K(\tau, y) f(y) dy. \tag{2.68}$$

Return now to the Parseval equality (2.49), which we spread for all functions $f(x) \in L_{1,2}(\mathbf{R}_+)$ and the corresponding Kontorovich-Lebedev transforms $K_{i\tau}[f] \in L_2\left(\mathbf{R}_+; \frac{2}{\pi^2} \tau \sinh(\pi\tau)\right)$. This fact provided by continuity of norms from the relation (see (2.53))

$$||f_n||_{L_{1,2}(\mathbf{R}_+)} = ||K_{i\tau}[f_n]||_{L_2\left(\mathbf{R}_+; \frac{2}{\pi^2}\tau \sinh(\pi\tau)\right)}. \tag{2.69}$$

Hence let us take in (2.48) $g(y) = 1, \ 0 < y \le x, \ g(y) = 0, y > x$. We arrive to the following formula

$$\int_0^x y f(y) dy = \frac{2}{\pi^2} \int_0^\infty \tau \sinh(\pi\tau) \int_0^x K_{i\tau}(u) du K_{i\tau}[f] d\tau. \tag{2.70}$$

Meanwhile, we apply formula 1.12.1.2 from Prudnikov et al. [2], which gives the calculation of the inner integral by u at the right-hand side of (2.70), namely

$$\int_0^x K_{i\tau}(u)du = \Re_{i\tau}\left[\frac{x^{1-i\tau}2^{i\tau}\Gamma(i\tau)}{1-i\tau}{}_1F_2\left(\frac{1-i\tau}{2};\ 1-i\tau,\ \frac{3-i\tau}{2};\frac{x^2}{4}\right)\right] \tag{2.71}$$

through the hypergeometric function ${}_1F_2(a;b,c;z)$ (see definition (1.45)). Thus carry out the differentiation and for almost all $x>0$ we obtain the dual formula for the inverse Kontorovich-Lebedev transform as

$$xf(x) = \frac{2}{\pi^2}\frac{d}{dx}\int_0^\infty \tau\sinh(\pi\tau)$$

$$\times\Re_{i\tau}\left[\frac{x^{1-i\tau}2^{i\tau}\Gamma(i\tau)}{1-i\tau}{}_1F_2\left(\frac{1-i\tau}{2};\ 1-i\tau,\ \frac{3-i\tau}{2};\frac{x^2}{4}\right)\right]K_{i\tau}[f]d\tau. \tag{2.72}$$

Prove now that formula (2.43) takes place, precisely we establish that the K-L transform $K_{i\tau}[f]$ is the limit in mean square by the norm of space $L_2\left(\mathbf{R}_+;\frac{2}{\pi^2}\tau\sinh(\pi\tau)\right)$ of the integral

$$\int_{1/N}^N K_{i\tau}(y)f(y)dy,$$

where $f(x)$ is an arbitrary function from the space $L_{1,2}(\mathbf{R}_+)$. For this in the equality

$$K_{i\tau}[f_N] = \frac{d}{d\tau}\int_0^\infty K(\tau,y)f_N(y)dy$$

$$= \frac{d}{d\tau}\int_{1/N}^N K(\tau,y)f(y)dy \tag{2.73}$$

perform the differentiation under the sign of integral due to uniform convergence of the differentiated integral. This leads us to the formula

$$K_{i\tau}[f_N] = \int_{1/N}^N K_{i\tau}(y)f(y)dy. \tag{2.74}$$

If now $K_{i\tau}[f]$ defined by formula (2.68), then the Parseval equality (2.49) provides the relation

$$||K_{i\tau}[f]-K_{i\tau}[f_N]||^2_{L_2\left(\mathbf{R}_+;\frac{2}{\pi^2}\tau\sinh(\pi\tau)\right)} = ||f-f_N||^2_{L_{1,2}(\mathbf{R}_+)}$$

$$= \int_{y\notin[1/N,N]} y|f(y)|^2dy \to 0,\ N\to\infty, \tag{2.75}$$

which means that $K_{i\tau}[f_N] \to K_{i\tau}[f]$ by norm of the space $L_2\left(\mathbf{R}_+;\frac{2}{\pi^2}\tau\sinh(\pi\tau)\right)$. Similarly we prove the convergence in mean of the sequence $\{f_N\}$ to f by the norm of $L_{1,2}$, if

$$f_N(x) = \frac{2}{\pi^2 x}\int_0^N \tau\sinh(\pi\tau)K_{i\tau}(x)K_{i\tau}[f]d\tau. \tag{2.76}$$

Thus we summarize our results in this section by the following *Plancherel theorem.*

Theorem 2.4. *The operator of the Kontorovich-Lebedev transform given by formula* (2.68) *maps the space* $L_{1,2}(\mathbf{R}_+)$ *onto the space* $L_2\left(\mathbf{R}_+; \frac{2}{\pi^2}\tau\sinh(\pi\tau)\right)$ *and the inverse operator described by relation* (2.72). *These operators are the limits in mean by the respective norm of Hilbert weighted spaces of integrals* (2.74), (2.76).

2.4 Composition representations of the K-L transform. A convolution Hilbert space

As we see the K-L transform (2.1) is quite different from the convolution type integral transforms as in view of the fact that its kernel is essentially a function of two variables. But nevertheless, we can derive a connection between K-L transform and on the other hand the Fourier, Laplace and Mellin transforms. Namely, from integral representation (1.98) substituting it into integral (2.1) for arbitrary function $f(x) \in L_{\nu,p}(\mathbf{R}_+)$, $\nu < 1, p \geq 1$ we obtain the estimate with the aid of the Hölder inequality (1.21) as (see also (2.4))

$$|K_{i\tau}[f]| \leq \frac{1}{2}\int_0^\infty |f(y)|dy \int_{-\infty}^\infty e^{-y\cosh u}du$$

$$= \int_0^\infty K_0(y)|f(y)|dy$$

$$\leq \left(\int_0^\infty K_0^q(y)y^{(1-\nu)q-1}dy\right)^{1/q}\left(\int_0^\infty |f(y)|^p y^{\nu p-1}dy\right)^{1/p}$$

$$= C||f||_{L_{\nu,p}(\mathbf{R}_+)}. \tag{2.77}$$

Hence the Fubini's theorem allows us to interchange the order of integration in the last iterated integral. As a result we arrive to the following composition

$$K_{i\tau}[f] = \sqrt{\frac{\pi}{2}}\left[F[Lf](\cosh u)\right](\tau) \tag{2.78}$$

in terms of the Fourier transform (1.191) and the Laplace transform (1.215) being calculated at the point $x = \cosh u$. All integrals in (2.78) are absolutely convergent. Also we assume in this case that $\tau \in \mathbf{R}$ which is possible in view of the evenness of the Macdonald function by its index. Moreover, with the generalized Minkowski inequality (1.10) one can easily conclude the belonging of $[Lf](\cosh u)$ to the space $L_q(\mathbf{R})$, $q \geq 1$, precisely

$$||[Lf](\cosh u)||_{L_q(\mathbf{R})} = \left(\int_{-\infty}^\infty \left|\int_0^\infty e^{-y\cosh u}f(y)dy\right|^p du\right)^{1/q}$$

$$\leq \int_0^\infty |f(y)|dy\left(\int_{-\infty}^\infty e^{-qy\cosh u}du\right)^{1/q} = 2^{1/q}\int_0^\infty K_0^{1/q}(qy)|f(y)|dy. \tag{2.79}$$

The Hölder inequality (1.21) gives us to continue the estimate (2.79) as

$$\int_0^\infty K_0^{1/q}(qy)|f(y)|dy \le \left(\int_0^\infty K_0(qy)y^{(1-\nu)q-1}dy\right)^{1/q} ||f||_{L_{\nu,p}(\mathbf{R}_+)}, \tag{2.80}$$

where $p = q/(q-1)$. But according to formula 2.16.2.2 from Prudnikov et al. [2] we can calculate the integral at the right-hand side of inequality (2.80) to obtain finally the estimate

$$||[Lf](\cosh u)||_{L_q(\mathbf{R})} \le 2^{\frac{2}{p}-\nu-1} q^{\nu-1} \Gamma^{\frac{2}{q}}\left(\frac{(1-\nu)q}{2}\right) ||f||_{L_{\nu,p}(\mathbf{R}_+)}. \tag{2.81}$$

This inequality can be used to estimate the L_p-norm of the K-L transform by Theorem 1.14 when $p \ge 2$. Namely, from (1.193) we find that

$$\left\|\sqrt{\frac{2}{\pi}}K_{i\tau}[f]\right\|^p_{L_p(\mathbf{R})} \le \frac{1}{(2\pi)^{\frac{q}{2}-1}}||[Lf](\cosh u)||^q_{L_q(\mathbf{R})}$$

$$\le \frac{2^{(1-\nu)q-1}q^{(\nu-1)q}}{\pi^{\frac{q}{2}-1}}\Gamma^2\left(\frac{(1-\nu)q}{2}\right) ||f||^q_{L_{\nu,p}(\mathbf{R}_+)}, \; q = p/(p-1). \tag{2.82}$$

The composition equality (2.78) is needed for further investigations of the mapping properties of the Kontorovich-Lebedev transform (2.1). However, we shall demonstrate now another composition representation of the Kontorovich-Lebedev transform involving the Mellin transform (1.203) as well as the Laplace transform (1.215) which is to be calculated at different from (2.78) point. This enable us to introduce the Kontorovich-Lebedev transform of an arbitrary complex index.

Actually, turning to the integral (1.238) write it in slightly different, but symmetric form, changing variable $u = \frac{x}{2t}$ and letting there an arbitrary complex index s instead of pure imaginary number $i\tau$. As result we find that

$$K_s(x) = \frac{1}{2}\int_0^\infty e^{-xh(u)}u^{s-1}du, \; x > 0, \tag{2.83}$$

where $h(u) = \frac{1}{2}(u + u^{-1})$. Hence immediately define the Kontorovich-Lebedev transform of an arbitrary complex index $s = \mu + i\tau$ as follows

$$K_s[f] = \int_0^\infty K_s(y)f(y)dy. \tag{2.84}$$

Appealing to the asymptotic behavior of the Macdonald function (1.97) and integral (2.83) deduce the inequality

$$|K_s(x)| \le K_{\Re s}(x), \; x > 0. \tag{2.85}$$

This estimate allows us to formulate the important result concerning analytic properties of the K-L transform (2.84) as a complex-valued function of variable s, mapping the Lebesgue space $L_{\nu,p}(\mathbf{R}_+)$ into the space of analytic functions at definite vertical strip. Thus our discussions at the beginning of the present chapter, concerning differential properties of the Kontorovich-Lebedev integral (2.1) at the real positive half axis should be naturally extended to the complex case.

Theorem 2.5. *Under conditions of Lemma* 2.1 *the Kontorovich-Lebedev transform $K_s[f]$ is analytic function at the open vertical strip $|\Re s| < 1-\nu$.*

Proof. Indeed, appealing to Lemma 2.1 and invoking with inequality (2.85) we obtain the uniform estimate like (2.4) with respect to an imaginary part of the variable s, namely

$$|K_s[f]| \le \int_0^\infty K_{\Re s}(y)|f(y)|dy$$
$$\le \left(\int_0^\infty K^q_{\Re s}(y)y^{(1-\nu)q-1}dy\right)^{1/q}\left(\int_0^\infty |f(y)|^p y^{\nu p-1}dy\right)^{1/p}$$
$$= C||f||_{L_{\nu,p}(\mathbf{R}_+)}, \tag{2.86}$$

where the integral

$$C = \left(\int_0^\infty K^q_{\Re s}(y)y^{(1-\nu)q-1}dy\right)^{1/q} < \infty \tag{2.87}$$

under condition $|\Re s| < 1-\nu$ which provides the uniform convergence of the Kontorovich-Lebedev integral (2.84) and consequently, its analyticity at this strip. Theorem 2.5 is proved. •

Corollary 2.2. *The Kontorovich-Lebedev transform* (2.84) *of an arbitrary complex index s of the function $f(x) \in L_{\nu,p}(\mathbf{R}_+)$, $p \ge 1, \nu < 1$ can be represented at the strip $|\Re s| < 1-\nu$ by the formula*

$$K_s[f] = \frac{1}{2}\mathcal{M}\left([Lf]\left(\frac{1}{2}\left(t+\frac{1}{t}\right)\right); s\right), \tag{2.88}$$

involving the Laplace transform (1.215) *at the point $h(t) = \frac{1}{2}(t+t^{-1})$ and the Mellin transform* (1.203).

Proof. From estimate (2.86) using integral (2.83) we immediately arrive to composition (2.88) by Fubini's theorem. •

To inverse the Kontorovich-Lebedev transform (2.84) of an arbitrary index we have to choose the respective space of functions, which describes its range. Following to Ditkin [1]-[2] we shall construct the so-called *convolution Hilbert spaces* of functions and apply theirs to integral representations, in particular, to the K-L transform (2.84).

Let $h(u)$ and $q(u)$ be positive functions on $\mathbf{R}_+$ such that the integral

$$\int_0^\infty e^{-xh(u)}q(u)du = \omega(x) \tag{2.89}$$

converges for all $x > 0$ and defines some weight function. Let us assume for arbitrary function $h(u)$ mentioned above that if

$$\int_0^\infty e^{-xh(u)}f(x)dx = 0 \tag{2.90}$$

for any number $u > 0$, then $f(x) = 0$ almost everywhere on $\mathbf{R}_+$.

Denote by S_0 the set of all absolutely integrable functions $f(x)$ defined on the domain $\mathbf{R}_+$ such that $f(x)$ is compactly supported in this domain. If $f(x) \in S_0$ and $g(x) \in S_0$, then as is known their *Laplace convolution*

$$(f * g)(x) = \int_0^x f(x-y)g(y)dy \tag{2.91}$$

belongs also to the space S_0. Therefore, the integral

$$\int_0^\infty (f * \overline{g})(x)\omega(x)dx = \langle f, g\rangle \tag{2.92}$$

exists. Hence on the linear set S_0 we can define operator of the Laplace transform (1.215) $[L_h f](u) \equiv [Lf](h(u)), f \in S_0$. Setting by $\varphi(u) = [L_h f](u)$ and by $\psi(u) = [L_h g](u)$ due to the known factorization property of the Laplace convolution (2.91) (see, for example Titchmarsh [1]) we obtain the following equality

$$\varphi(u)\overline{\psi(u)} = \int_0^\infty (f * \overline{g})(x)e^{-xh(u)}dx = [L_h(f * \overline{g})](u). \tag{2.93}$$

Condition (2.89) and Fubini's theorem give us the convergence of the integral

$$\int_0^\infty \varphi(u)\overline{\psi(u)}q(u)du$$

and the equality

$$\int_0^\infty \varphi(u)\overline{\psi(u)}q(u)du = \int_0^\infty (f * \overline{g})(x)\omega(x)dx = \langle f, g\rangle. \tag{2.94}$$

It follows from (2.94) and condition (2.90) that (2.92) has all properties of an inner product. The set S_0 becomes a pre-Hilbert space with this inner product. Its completion is called a *convolution Hilbert space* and is denoted by $S(q)$. Thus, an inner product $\langle f, g\rangle$ and a norm

$$||f||_S = \sqrt{\langle f, f\rangle}$$

are defined for any elements $f, g \in S_0$. If $f, g \in S_0$, then as is obvious

$$\langle f, g\rangle = \int_0^\infty (f * \overline{g})(x)\omega(x)dx$$

$$= \int_0^\infty \int_0^\infty f(x)\overline{g(y)}\omega(x+y)dxdy. \tag{2.95}$$

The last equality implies that if $f(x)$ satisfies the condition

$$\int_0^\infty \int_0^\infty |f(x)f(y)|\omega(x+y)dxdy < \infty, \tag{2.96}$$

then $f \in S(q)$. Further, if $f(x)$ and $g(x)$ satisfy (2.96), then the Cauchy-Schwarz-Bunyakovskii inequality gives us a convergence of the integral

$$\int_0^\infty \int_0^\infty |f(x)g(y)|\omega(x+y)dxdy \tag{2.97}$$

and the equality

$$\langle f, g\rangle = \int_0^\infty (f * \overline{g})(x)\omega(x)dx. \tag{2.98}$$

Let us consider now the weighted Hilbert space $L_2(\mathbf{R}_+; q(u))$ formed by the functions $\varphi(u),\ u \in \mathbf{R}_+$ such that

$$\int_0^\infty |\varphi(u)|^2 q(u)du < \infty.$$

The inner product here is

$$(\varphi, \psi) = \int_0^\infty \varphi(u)\overline{\psi(u)}q(u)du$$

and the norm is

$$||\varphi|| = \sqrt{(\varphi, \varphi)}. \tag{2.99}$$

As we have shown (see (2.94)), the operator $[L_h f]$ maps the set S_0 into the space $L_2(\mathbf{R}_+; q(u))$, and for all $f \in S_0$

$$||[L_h f]||^2 = \int_0^\infty |\varphi(u)|^2 q(u)du$$

$$= \int_0^\infty (f * \overline{f})(x)\omega(x)dx = ||f||_S^2. \tag{2.100}$$

The extension of $[L_h f]$ by continuity to the whole space $S(q)$ is denoted by $[L_h^q f]$. Thus, $[L_h^q f]$ is defined for all $f \in S(q)$, its range $L_h^q S$ belongs to $L_2(\mathbf{R}_+; q(u))$, and for all $f \in S(q)$

$$||f||_S = ||[L_h^q f]||, \tag{2.101}$$

and $[L_h^q f] = 0$ if and only if $f = 0$. Consequently, the inverse operator exists and is bounded.

Let us return now to the Kontorovich-Lebedev transform (2.84) with the Macdonald function (2.83) and $h(u) = \frac{1}{2}(u + u^{-1})$. As far as the Corollary 2.2 is concerned, from composition (2.88) we obtain that for $\varphi = [L_h^q f]$ is obviously $\varphi(u) = \varphi(1/u)$ for any $u > 0$. Consequently, the range $L_h^q S$ does not coincide with $L_2(\mathbf{R}_+; q(u))$. The set of functions satisfying $\varphi(u) = \varphi(1/u)$ is a subspace of $L_2(\mathbf{R}_+; q(u))$. We prove that this subspace coincides with $L_h^q S$. Indeed, otherwise there would exist a nonzero function $\varphi_0(u)$ such that $\varphi_0(u) = \varphi_0(1/u)$ for any $u > 0$ and φ_0 is orthogonal to $L_h^q S$. But this cannot be, because $(\varphi_0, [L_h^q f]) = 0$ for any $f \in S(q)$ implies that in particular

$$\int_0^\infty \varphi_0(u)e^{-ah(u)}q(u)du = 0 \tag{2.102}$$

for all $a > 0$ if we establish that the function $e^{-ah(u)}$ is an element of the space $L_h^q S$. If it is true then from the properties of the Laplace transform (1.215) we lead to conclusion that $\varphi_0(u) = 0$ for $1 < u < \infty$, and hence for all $u > 0$ according to analytic properties of the Laplace transform (1.215).

Concerning the fact that $e^{-ah(u)} \in L_h^q S$ for any $a > 0$ let us observe first that from (2.89) it follows the conclusion $e^{-ah(u)} \in L_2(\mathbf{R}_+; q(u))$. Let now $f_n(x) = 0$ for

$0 < x < a$ or $x > a + 1/n$ and $f_n = n$ for $a \le x \le a + 1/n$, $n = 1, 2, \dots$. Then it is not difficult to calculate the corresponding image of the Laplace transform, namely

$$\varphi_n(u) = [L_h^q f_n]) = e^{-ah(u)} n \frac{1 - \exp\left(-\frac{h(u)}{n}\right)}{h(u)}, \tag{2.103}$$

which implies that

$$\lim_{n\to\infty} ||e^{-ah(u)} - \varphi_n(u)|| = 0. \tag{2.104}$$

Consequently, $e^{-ah(u)} \in L_h^q S$. Therefore, there exists in $S(q)$ an element, namely delta-function $\delta(x, a)$ such that $[L_h^q \delta](u) = e^{-ah(u)}$.

Thus we proved that the set of Laplace transforms $\varphi(u) = [L_h^q f](u) = \varphi(1/u)$ coincides with the range $L_h^q S$. Since $h(u) = \frac{1}{2}(u + u^{-1}) \ge 1$ for any $u > 0$, we have naturally that

$$||\varphi(u)/h(u)|| \le ||\varphi||$$

for any function $\varphi(u) \in L_2(\mathbf{R}_+; q(u))$. If $\varphi \in L_h^q S$, then $\varphi(u)/h(u) \in L_h^q S$. Furthermore, if $f \in S(q)$ is a regular function, then integration by parts it is not difficult to motivate the fact that

$$\left[L_h^q \int_0^x f(y)dy\right](u) = \int_0^\infty e^{-\frac{x}{2}(u+u^{-1})}dx \int_0^x f(y)dy = \frac{2\varphi(u)}{u + u^{-1}}. \tag{2.105}$$

Choose now concrete weight function $q(u) = u^{2\sigma-1}$, $\sigma \in \mathbf{R}$ and we obtain the Hilbert space $L_{\sigma,2}(\mathbf{R}_+)$. So if some function $\varphi(u) \in L_{\sigma,2}(\mathbf{R}_+)$, then according to Theorem 1.15 its Mellin's transform (1.203) $\varphi^*(s)$ exists and belongs to the space $L_2(\sigma - i\infty, \sigma + i\infty)$ and

$$\varphi^*(s) = \text{l.i.m.}_{N\to\infty} \int_{1/N}^N \varphi(u)u^{s-1}du. \tag{2.106}$$

Inversely we have formula like (1.204)

$$\varphi(x) = \frac{1}{2\pi i}\text{l.i.m.}_{N\to\infty} \int_{\sigma-i\infty}^{\sigma+i\infty} \varphi^*(s)x^{-s}ds. \tag{2.107}$$

Calculating integral (2.89) for our case of functions $h(u)$ and $q(u)$ use formula (2.83) and immediately obtain $\omega(x) = 2K_{2\sigma}(x)$, $x > 0$. Hence turn to the equality (see (2.96))

$$\begin{aligned} &2\int_0^\infty \int_0^\infty |f(x)f(y)|K_{2\sigma}(x+y)dxdy \\ &= \int_0^\infty u^{2\sigma-1}du \left(\int_0^\infty e^{-yh(u)}|f(y)|dy\right)^2, \end{aligned} \tag{2.108}$$

provided that $f(x) \in L_{\nu,p}$, $p \ge 1, \nu < 1$, which is obtained by Fubini's theorem. Indeed, invoking with generalized Minkowski's inequality (1.10) and Hölder's inequality (1.21) we have the estimate

$$\left(\int_0^\infty u^{2\sigma-1}du \left(\int_0^\infty e^{-yh(u)}|f(y)|dy\right)^2\right)^{1/2}$$

$$\leq \int_0^\infty |f(y)|dy \left(\int_0^\infty u^{2\sigma-1}e^{-2yh(u)}du\right)^{1/2} = \sqrt{2}\int_0^\infty K_{2\sigma}^{1/2}(2y)|f(y)|dy$$

$$\leq \sqrt{2}||f||_{\nu,p}\left(\int_0^\infty K_{2\sigma}^{q/2}(2y)y^{(1-\nu)q-1}dy\right)^{1/q} < \infty,\ q = p/(p-1) \tag{2.109}$$

under conditions $f(x) \in L_{\nu,p}$, $p \geq 1, \nu < 1, |\sigma| < 1-\nu$. Hence we find from inequality (2.96) that this function $f(x)$ is an element of the respective convolution Hilbert space $S(q)$ (2.98) with the inner product as

$$\langle f, g\rangle = 2\int_0^\infty (f * \overline{g})(x)K_{2\sigma}(x)dx,\ |\sigma| < 1-\nu. \tag{2.110}$$

Corollary 2.2, precisely equality (2.88) enables us to put $\varphi^*(s) = 2K_s[f]$. Moreover, since for the Kontorovich-Lebedev transform the inside Laplace transform in composition (2.88) satisfies the condition $\varphi(u) = \varphi(1/u)$ for any $u > 0$ from equality (2.107) we obtain the related one as

$$\varphi(x) = \frac{1}{2\pi i}\text{l.i.m.}_{N\to\infty}\int_{\sigma-iN}^{\sigma+iN}\varphi^*(s)x^s ds, \tag{2.111}$$

where the integral (2.111) converges in $L_{-\sigma,2}(\mathbf{R}_+)$. This implies the following equality

$$\frac{\varphi(x)}{h(x)} = \frac{1}{2\pi i}\text{l.i.m.}_{N\to\infty}\int_{\sigma-iN}^{\sigma+iN}\varphi^*(s)\frac{x^{-s}+x^s}{2h(x)}ds. \tag{2.112}$$

Meanwhile, the function $(x^{-s}+x^s)/h(x)$ belongs to the range of convolution Hilbert space $S(q)$ by means of the Laplace transform $\varphi(x) = [L_h^q f](x)$ with $h(u) = \frac{1}{2}(u + u^{-1})$, $q(u) = u^{2\sigma-1}$ if

$$\int_0^\infty \left|\frac{u^{-s}+u^s}{u+u^{-1}}\right|^2 u^{2\sigma-1}du < \infty. \tag{2.113}$$

From the inequality

$$\int_0^\infty \left|\frac{u^{-s}+u^s}{u+u^{-1}}\right|^2 u^{2\sigma-1}du \leq \int_0^\infty \left(\frac{u^{2\sigma}+1}{u^2+1}\right)^2 udu < \infty,\ |\sigma| < 1/2 \tag{2.114}$$

the desired result has been satisfied. Considering equality (2.105), we achieve that the function

$$\int_0^x f(y)dy \in L_{\nu-1,p}(\mathbf{R}_+),$$

if $f(x) \in L_{\nu,p}(\mathbf{R}_+), \nu < 1$. Actually it follows from estimate

$$\left(\int_0^\infty x^{(\nu-1)p-1}dx\left|\int_0^x f(y)dy\right|^p\right)^{1/p} = \left(\int_0^\infty x^{\nu p-1}dx\left|\int_0^1 f(xy)dy\right|^p\right)^{1/p}$$

$$\leq ||f||_{\nu,p}\int_0^1 y^{-\nu}dy < \infty,\ \nu < 1. \tag{2.115}$$

Thus if we choose $|\sigma| < \min(1/2, 1-\nu)$ from inequalities (2.109), (2.115) we obtain that functions $f(x)$ and $\int_0^x f(y)dy$ are elements of the convolution Hilbert space

(2.110). The respective Laplace transforms $\varphi(x)$ and $\varphi(x)/h(x)$ belong to the range $L_h^q S$.

Making use integral 2.16.6.1 from Prudnikov et al. [2] it is not difficult to show that for any complex $s = \sigma + i\tau$ with $|\sigma| < \min(1/2, 1-\nu)$ the following formula is true

$$\frac{x^{-s} + x^{s}}{x + x^{-1}} = \int_0^\infty \left(\frac{s \sin(\pi s)}{\pi} \int_u^\infty \frac{K_s(t)}{t} dt + \cos\left(\frac{\pi s}{2}\right)\right) e^{-u(x+x^{-1})} du$$

$$= \left[L_h^q \left(\frac{s \sin(\pi s)}{\pi} \int_u^\infty \frac{K_s(t)}{t} dt + \cos\left(\frac{\pi s}{2}\right)\right)\right](x). \tag{2.116}$$

On the other hand, substituting it in (2.112) and invoking with relation (2.105) we find the following equality

$$\left[L_h^q \int_0^u f(y) dy \right](x) = \frac{1}{2\pi i} \text{l.i.m.}_{N\to\infty} \int_{\sigma - iN}^{\sigma + iN} \varphi^*(s)$$

$$\times \left[L_h^q \left(\frac{s \sin(\pi s)}{\pi} \int_u^\infty \frac{K_s(t)}{t} dt + \cos\left(\frac{\pi s}{2}\right)\right)\right](x) ds. \tag{2.117}$$

However, according to Theorem 2.5 the K-L transform (2.84) $K_s[f] = \varphi^*(s)/2$ is an analytic function at the strip $|\sigma| < 1 - \nu$ and therefore one can interchange the order of integration at the right-hand side of equality (2.117) which gives

$$\left[L_h^q \int_0^u f(y) dy \right](x) = \left[L_h^q \frac{1}{\pi^2 i} \text{l.i.m.}_{N\to\infty} \int_{\sigma - iN}^{\sigma + iN} K_s[f] \right.$$

$$\left. \times \left(s \sin(\pi s) \int_u^\infty \frac{K_s(t)}{t} dt + \cos\left(\frac{\pi s}{2}\right)\right) ds \right](x). \tag{2.118}$$

Since the linear operator $[L_h^q f]$ is bounded on the convolution space $S(q)$ due to the norm equality (2.101) we can omit it and deduce that

$$\int_0^x f(y) dy = \frac{1}{\pi^2 i} \text{l.i.m.}_{N\to\infty} \int_{\sigma - iN}^{\sigma + iN} K_s[f]$$

$$\times \left(s \sin(\pi s) \int_x^\infty \frac{K_s(t)}{t} dt + \cos\left(\frac{\pi s}{2}\right)\right) ds, \ x > 0, \tag{2.119}$$

where the convergence of the integral in (2.119) means by norm (2.100) defined by the inner product (2.110). Obviously, if $f(x) \in L_{\nu,p} \subset S(q)$ under conditions $p \geq 1, \nu < 1$ then $f(x) \in L_1([0,x])$ and one can differentiate the left-hand side of equality (2.119) almost everywhere. Meanwhile, for the Macdonald function (2.83) by the interchange $u = e^v$ we arrive to the respective integral representation (1.99) as

$$K_s(x) = \frac{1}{2} \int_{i\delta - \infty}^{i\delta + \infty} e^{-x \cosh v + sv} dv, \ x > 0, \ \delta \in [0, \pi/2). \tag{2.120}$$

Hence we immediately obtain the uniform estimate like (1.100), precisely

$$|K_s(x)| \leq e^{-\delta|\tau|} K_\sigma(x \cos\delta), \ s = \sigma + i\tau, \tau \in \mathbf{R}. \tag{2.121}$$

Thus if $K_{\sigma+i\tau}[f] \in L_1(\mathbf{R}; \tau e^{\tau(\pi-\delta)})$, $\delta \in [0, \pi/2)$, then owing to estimate

$$\int_{-\infty}^{\infty} \left| K_{\sigma+i\tau}[f](\sigma + i\tau) \sin(\pi(\sigma + i\tau)) \int_x^{\infty} \frac{K_{\sigma+i\tau}(t)}{t} dt d\tau \right|$$

$$\leq C \int_{-\infty}^{\infty} |K_{\sigma+i\tau}[f]| \, \tau e^{\tau(\pi-\delta)} d\tau \int_x^{\infty} \frac{K_\sigma(t)}{t} dt < \infty, \; x > 0, \tag{2.122}$$

where C is a positive constant we conclude that the integral in the right-hand side of equality (2.119) is absolutely convergent and moreover, we perform the differentiation under its sign. So we lead to the inversion formula for the Kontorovich-Lebedev transform (2.84) in the form

$$f(x) = \frac{i}{\pi^2} \int_{\sigma-i\infty}^{\sigma+i\infty} s \sin(\pi s) \frac{K_s(x)}{x} K_s[f] ds, \tag{2.123}$$

which is valid for almost all $x \in \mathbf{R}_+$ and is natural generalization of the inversion formula for the K-L transform for $\sigma = 0$. Thus we established the following final theorem.

Theorem 2.6. *Let $f(x)$ be from the space $L_{\nu,p}(\mathbf{R}_+)$ provided that $p \geq 1, \nu < 1$. Then the Kontorovich-Lebedev transform (2.84) $K_s[f]$, $s = \sigma + i\tau$ exists and belongs to the space $L_2(\sigma - i\infty, \sigma + i\infty)$, $|\sigma| < \min(1/2, 1 - \nu)$. Moreover, almost for all $x \in \mathbf{R}_+$ inversion formula (2.119) is true, where the convergence of the integral is meant by the norm generated by the convolution Hilbert space (2.110). If besides $K_{\sigma+i\tau}[f] \in L_1(\mathbf{R}; \tau e^{\tau(\pi-\delta)})$, $\delta \in [0, \pi/2)$, then almost everywhere on $\mathbf{R}_+$ the representation (2.123) holds.*

2.5 Representations through the Mellin transform. Watson's type lemma

In this section we give another representation of the Kontorovich-Lebedev transform (2.1) generated by the Mellin-Parseval equality (1.214). Such approach enables us to investigate the composition structure of the K-L transform and gives a method of the various other constructions of index transforms and their inversions. These questions were already considered recently by the author in Vu Kim Tuan et al. [1], Yakubovich [1], [3]-[4], Samko et al. [1], Yakubovich and Luchko [2] for the so-called *Kontorovich-Lebedev type index transforms* in special functional spaces that account asymptotic behavior of the gamma-functions at infinity. Here we attract our attention to L_p-theory of these mentioned transforms and first describe the corresponding properties for the K-L transform (2.1).

Let us consider the Mellin-Barnes integral (1.59) for Meijer's G-function related to the Macdonald function $K_{i\tau}(2\sqrt{y})$, $y > 0$, namely formula (1.113). Hence after changing the variable $2\sqrt{y} = x$ this easily yields to the following relation

$$K_{i\tau}(x) = \frac{1}{4\pi i} \int_{\nu-i\infty}^{\nu+i\infty} 2^{s-1} \Gamma\left(\frac{s+i\tau}{2}\right) \Gamma\left(\frac{s-i\tau}{2}\right) x^{-s} ds, \ x > 0, \tag{2.124}$$

where the variable s lies along the vertical contour $s = \nu + it$ with $\nu > 0,\ t \in \mathbf{R}$. The reciprocal Mellin formula (1.203) in this case becomes as

$$\int_0^\infty K_{i\tau}(y) y^{s-1} dy = 2^{s-2} \Gamma\left(\frac{s+i\tau}{2}\right) \Gamma\left(\frac{s-i\tau}{2}\right), \ \Re s > 0, \tag{2.125}$$

(see formula 2.16.2.2 from Prudnikov et al. [2]). Further, all series of poles of the gamma-functions in (2.124) are separated by this contour, and Stirling's formula (1.32) gives their asymptotic behavior at infinity for each index $\tau \geq 0$ as

$$\left| \Gamma\left(\frac{s+i\tau}{2}\right) \Gamma\left(\frac{s-i\tau}{2}\right) \right| = O\left(e^{-\frac{\pi}{2}|t|} |t|^{\nu-1}\right). \tag{2.126}$$

Therefore we easily conclude that integral (2.124) is absolutely convergent and moreover its integrand belongs to any space $L_p(\nu - i\infty, \nu + i\infty),\ p \geq 1$. Turning now to Theorem 1.17 that allows us to write for the K-L transform (2.1) the Parseval equality (1.214), considering the Kontorovich-Lebedev transform for each fixed index τ as the Mellin convolution type integral transform (1.220) at the point $x = 1$. Precisely, if $f(x) \in L_{\nu,p}(\mathbf{R}_+)$ for $1 < p \leq 2$, then invoking with relation (2.124) we obtain the formula

$$K_{i\tau}[f] = \frac{1}{4\pi i} \int_{\nu-i\infty}^{\nu+i\infty} 2^{-s} \Gamma\left(\frac{1-s+i\tau}{2}\right) \Gamma\left(\frac{1-s-i\tau}{2}\right) f^*(s) ds, \ \nu < 1, \tag{2.127}$$

where $f^*(s) \equiv f^*(\nu + it)$ is the Mellin transform of the function $f(x)$ and it is from the space $L_q(\mathbf{R}),\ q = p/(p-1)$ according to Theorem 1.15. Thus we established the following result.

Theorem 2.7. *Let $f(x)$ be from the space $f(x) \in L_{\nu,p}(\mathbf{R}_+),\ 1 < p \leq 2,\ \nu < 1$. Then the K-L transform* (2.1) *can be represented by formula* (2.127) *and both of integrals are absolutely convergent.*

Note that integral (2.1) is absolutely convergent due to Lemma 2.1 and the absolute convergence of integral (2.127) can be easily verified by the Hölder inequality.

Now we briefly concern the question of generalization the Kontorovich-Lebedev transform on the index transforms with different hypergeometric type of special functions as the kernels that have been considered in Chapter 1. Namely, using the list of formulae (1.107)-(1.140) of particular cases of Meijer's G-function, which corresponds by the structure to index kernels, one can introduce index transforms by formula (2.127). Some of theirs are familiar and shall be considered in detail in the next chapters. We can construct also the new operators basing on the L_p-theory of the Kontorovich-Lebedev transform and Mellin convolution type transforms.

Let us substitute into formula (2.127) instead of the function $f^*(s)$ the product $\theta^*(s) f^*(s)$, where the function $\theta^*(s)$ is fixed and in most cases, especially for the index transforms constructions it is gamma-ratios like formulae (1.60), (1.64). If $\theta^*(\nu + it) \in L_p\left(\mathbf{R}; e^{-\frac{\pi}{2}|t|}|t|^{-\nu}\right)$, $1 < p \leq 2$, then denote by $Y_{i\tau}^{\theta}(x)$ the function

$$Y_{i\tau}^{\theta}(x) = \frac{1}{4\pi i}\int_{1-\nu-i\infty}^{1-\nu+i\infty} 2^{s-1}\Gamma\left(\frac{s+i\tau}{2}\right)\Gamma\left(\frac{s-i\tau}{2}\right)\theta^*(1-s)x^{-s}ds, \ \nu < 1, \quad (2.128)$$

where the integral (2.128) is convergent at least by the norm $L_{1-\nu,q}(\mathbf{R}_+)$. Hence we introduce the new index transform as follows

$$Y_{i\tau}^{\theta}[f] = \frac{1}{4\pi i}\int_{\nu-i\infty}^{\nu+i\infty} 2^{-s}\Gamma\left(\frac{1-s+i\tau}{2}\right)\Gamma\left(\frac{1-s-i\tau}{2}\right)\theta^*(s)f^*(s)ds$$

$$= \int_0^{\infty} Y_{i\tau}^{\theta}(t)f(t)dt. \quad (2.129)$$

Obviously, we used here the Mellin-Parseval equality (1.214). The general index transform (2.129) comprises a wide set of familiar examples as well as new ones. For instance, taking formula (1.117) after simple interchange of variable and invoking with the Mellin transform property (1.207) write the respective integral (2.128) as

$$Y_{i\tau}^{\theta}(x) = \frac{1}{4\pi i}\int_{1-\nu-i\infty}^{1-\nu+i\infty} 2^{s-1}\Gamma\left(\frac{s+i\tau}{2}\right)\Gamma\left(\frac{s-i\tau}{2}\right)$$

$$\times\frac{\Gamma(s/2)}{\Gamma((s+1)/2)}x^{-s}ds = \frac{1}{\sqrt{\pi}}K_{i\tau/2}^2\left(\frac{x}{2}\right). \quad (2.130)$$

Here $\theta^*(s) = \Gamma((1-s)/2)/\Gamma(1-s/2)$ and as we can verified by Stirling's formula (1.32), this function satisfies the above condition $\theta^*(\nu + it) \in L_p\left(\mathbf{R}; e^{-\frac{\pi}{2}|t|}|t|^{-\nu}\right)$, $1 < p \leq 2$. Hence we obtain the index transform with the square of the Macdonald function first introduced by Lebedev [8], namely

$$K_{i\tau/2}^2[f] = \frac{1}{\sqrt{\pi}}\int_0^{\infty} K_{i\tau/2}^2\left(\frac{y}{2}\right) f(y)dy. \quad (2.131)$$

Similarly we can write other examples of the index transforms and shall do it later. Now it is important to note the fact of connection between the K-L transform, general index transforms and the Mellin convolution type integral transforms. Actually, we can express general transform (2.129) through the K-L transform (2.1) by means the same Mellin-Parseval equality (1.214). Let us denote by

$$[\Theta f](x) = \frac{1}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty} \theta^*(s)f^*(s)x^{-s}ds \quad (2.132)$$

the operator being connected with the inverse Mellin transform (1.204), meaning that the integral (2.132) is convergent by $L_{\nu,p}$-norm. Then applying Theorem 1.17 we immediately obtain the composition equality as

$$Y_{i\tau}^{\theta}[f] = K_{i\tau}[[\Theta f]]. \quad (2.133)$$

Conversely, as is known operator (2.132) by the Mellin-Parseval equality (1.214) becomes the Mellin convolution type operator like (1.220) provided that there exists the inverse Mellin transform (1.204) of the function $\theta^*(s)$ at the respective sense. Although this condition is not necessary for existence the general index transform (2.129), because integral (2.128) can be convergent in spite of the fact, that integral (1.204) for $\theta^*(s)$ remains divergent one.

Theorem 2.8. *Let $f(x) \in L_{\nu,p}(\mathbf{R}_+)$, $1 < p \leq 2$, $\nu < 1$. Let also $\theta(s)$ be a complex valued function defined on the line $s = \nu + it$, $t \in \mathbf{R}$. If $\theta^*(\nu+it) \in L_p\left(\mathbf{R}; e^{-\frac{\pi}{2}|t|}|t|^{-\nu}\right)$ then the general index transform $Y^{\theta}_{i\tau}(x)$ exists by formula* (2.129) *with the respective index kernel* (2.128). *Besides the composition representation* (2.133) *is true, where the operator $[\Theta f]$ is defined by formula* (2.132).

Our purpose now is to consider the analog of the familiar Watson lemma concerning asymptotic expansion of the Laplace transform like (1.215) for analytic functions at the neighborhood of infinity (see details, for example, in Olver [1]). As is turned out to be this analog of the Watson lemma is valid for the K-L transform (2.1), precisely one can obtain its asymptotic when $\tau \to +\infty$ for functions $f(x)$ associated with functions of the exponential type.

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an analytic function in closed circle $|z| \leq r$. Then as is known for instance in Titchmarsh [2], by Cauchy's inequality for Taylor's coefficients we obtain the uniform estimate $|a_n r^n| < M$, where M is independent of n. Hence, if we denote by

$$\eta(z) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma^2((n+1)/2)} \left(\frac{z}{2}\right)^n \tag{2.134}$$

the function associated with $f(z)$, then we have the inequality

$$|\eta(z)| < M \sum_{n=0}^{\infty} \frac{|z|^n}{(2r)^n \Gamma^2((n+1)/2)}. \tag{2.135}$$

Meanwhile, due to the Stirling formula (1.32)

$$\begin{aligned} \Gamma^2\left(\frac{n+1}{2}\right) &= \frac{2\pi}{e}\left(\frac{n+1}{2e}\right)^n [1 + O(1/n)] \\ &= \sqrt{\frac{2\pi}{n}} \frac{\left(n+\frac{1}{n}\right)^n}{e} \frac{n!}{2^n}[1 + O(1/n)], \ n \to \infty. \end{aligned} \tag{2.136}$$

Hence,

$$|\eta(z)| < C \sum_{n=0}^{\infty} \left(\frac{n^{1/(2n)}}{r}\right)^n \frac{|z|^n}{n!} < C \sum_{n=0}^{\infty} \left(\frac{\xi}{r}\right)^n \frac{|z|^n}{n!} = Ce^{|z|\xi/r}, \tag{2.137}$$

where there exists some constant ξ such that $1 < n^{1/(2n)} < \xi < 2$. Thus the function $\eta(z)$ is exponential type function. Let us show that the K-L transform (2.1) of function $\eta(y), y \in \mathbf{R}_+$ exists under condition $r > \xi$. Indeed, invoking with inequality (1.147) we obtain

$$|K_{i\tau}[\eta]| \leq \int_0^{\infty} K_0(y)|\eta(y)|dy$$

$$< C\int_0^\infty K_0(y)e^{\alpha y}dy < \infty, \ \alpha = \xi/r < 1, \tag{2.138}$$

and the convergence of the last integral is motivated by asymptotic formulae (1.96)-(1.97) of the Macdonald function. Therefore, substituting series (2.134) into formula (2.1) perform to change the order of integration and summation which gives

$$\int_0^\infty K_{i\tau}(y)dy \sum_{n=0}^\infty \frac{a_n}{\Gamma^2((n+1)/2)}(2y)^n = \sum_{n=0}^\infty \frac{a_n}{2^n\Gamma^2((n+1)/2)}\int_0^\infty K_{i\tau}(y)y^n dy. \tag{2.139}$$

The inner integral can be calculated by formula (2.125) and we have the result

$$\int_0^\infty K_{i\tau}(y)y^n dy = 2^{n-1}\left|\Gamma\left(\frac{n+1+i\tau}{2}\right)\right|^2. \tag{2.140}$$

So the final equality from (2.139) becomes as

$$K_{i\tau}[\eta] = \frac{1}{2}\sum_{n=0}^\infty \frac{a_n}{\Gamma^2((n+1)/2)}\left|\Gamma\left(\frac{n+1+i\tau}{2}\right)\right|^2. \tag{2.141}$$

The last series can be written as follows

$$\begin{aligned} K_{i\tau}[\eta] &= \frac{1}{2}\left(\sum_{n=0}^{N} + \sum_{n=N}^{\infty}\right)\frac{a_n}{\Gamma^2((n+1)/2)}\left|\Gamma\left(\frac{n+1+i\tau}{2}\right)\right|^2 \\ &= S_N(\tau) + R_N(\tau). \end{aligned} \tag{2.142}$$

The remainder term $R_N(\tau)$ can be estimated uniformly by $\tau \geq 0$ applying inequality (1.26), namely

$$|R_N(\tau)| \leq \frac{1}{2}\sum_{n=N}^\infty |a_n| < M\sum_{n=N}^\infty r^{-n} = \frac{M}{r^N}\sum_{n=0}^\infty r^{-n}, \tag{2.143}$$

where $r > \xi > 1$ by the assumption above. Therefore, $R_N(\tau) \to 0, \ N \to \infty$ uniformly for all $\tau \geq 0$. Choose and settle some enough large number N. Appealing then to asymptotic formula (1.33) we write that

$$\left|\Gamma\left(\frac{n+1+i\tau}{2}\right)\right|^2 = 2\pi\left(\frac{\tau}{2}\right)^n e^{-\frac{\pi}{2}\tau}[1 + O(1/\tau)], \ \tau \to +\infty. \tag{2.144}$$

Hence

$$S_N(\tau) = \pi e^{-\frac{\pi}{2}\tau}\sum_{n=0}^N \frac{a_n}{\Gamma^2((n+1)/2)}\left(\frac{\tau}{2}\right)^n [1 + O(1/\tau)], \ \tau \to +\infty, \tag{2.145}$$

and it tends to $\pi e^{-\frac{\pi}{2}\tau}\eta(\tau)$. Thus finally we obtain that

$$K_{i\tau}[\eta] \sim \pi e^{-\frac{\pi}{2}\tau}\eta(\tau) \tag{2.146}$$

as $\tau \to +\infty$ and the following analog of the Watson lemma for the Kontorovich-Lebedev transform (2.1) is true.

Lemma 2.6. *Let $\eta(z)$ be a complex valued function defined by series* (2.134), *which associated with an analytic in the closed circle $|z| \leq r$, $r > \xi > 1$ function $f(z)$. Then asymptotic equality* (2.146) *holds as $\tau \to +\infty$.*

We can consider instead of function $\eta(z)$ familiar *Borel's function*

$$\phi(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} \left(\frac{z}{2}\right)^n \tag{2.147}$$

associated with an analytic function $f(z)$ in the circle $|z| \leq r$ by means of the Laplace transform (see, for example Akhiezer [1]). In this case as one can see from Stirling's formula for factorials and by using the above estimates the function $\phi(z)$ also satisfies the asymptotic equality (2.146).

The Watson type Lemma 2.6 for the K-L transform (2.1) can be applied to obtain the asymptotic solution of the following homogeneous integral equation of the second kind

$$\frac{e^{\frac{\pi}{2}\tau}}{\pi} \int_0^{\infty} K_{i\tau}(y)\varphi(y)dy = \varphi(\tau). \tag{2.148}$$

It should be pointed out that the solution of integral equations of the second kind with index operators in the closed form is an open problem. Nevertheless, we can seek the asymptotic solution in the case of equation (2.148) near infinity if the asymptotic expansions of f near this point is known. We note that the uniqueness of an asymptotic solution follows from the uniqueness property of the asymptotic expansion of the given function with respect to the given asymptotic sequence. However, the existence of the solution itself does not follow in general from the existence of its asymptotic solution.

Thus from Lemma 2.6 it follows that the asymptotic solution of the (2.148) is representable as

$$\varphi(y) \sim \sum_{n=0}^{\infty} b_n \left(\frac{y}{2}\right)^n, \tag{2.149}$$

as $y \to +\infty$ and the coefficients b_n, $n = 0, 1, \ldots$ such that $b_n = a_n \Gamma^2((n+1)/2)$, where a_n, $n = 0, 1, \ldots$ are the coefficients of analytic in some closed circle $|z| \leq r$ function $f(z)$ associated with $\varphi(z)$.

2.6 The index-convolution Kontorovich-Lebedev transform

In this last section of the present chapter we introduce the integral transform with the kernel as the Macdonald function (1.91) $K_{i\tau}(xy)$ of three independent variables x, y, τ

$$KL[f](\tau, x) \equiv g(\tau, x) = \int_0^{\infty} K_{i\tau}(xy) f(y)dy, \tag{2.150}$$

which has been announced first in Yakubovich [10] and we called it *the index-convolution Kontorovich-Lebedev transform.* Here as usually $x, y, \tau \in \mathbf{R}_+$, $f(y)$ is arbitrary measurable function from the respective Lebesgue space. The transform (2.150) is the function $g(\tau, x)$ of two variables $(\tau, x) \in \mathbf{R}_+ \times \mathbf{R}_+$ and it maps functions from one-dimensional into two-dimensional Lebesgue spaces. As is known when we fixed the parameter τ, we obtained Meijer's transform like (1.220) of the Mellin convolution type (see Zemanian [1], Prudnikov et al. [5]). Otherwise, when the parameter x is fixed we have the usual K-L transform like (2.1) the inversion formula of which can be written formally following to expansion (1.231) as

$$yf(y) = \frac{2}{\pi^2} \int_0^\infty \tau \sinh(\pi\tau) K_{i\tau}(xy) KL[f](\tau, x) d\tau. \tag{2.151}$$

For our further consideration let us denote by $L_{\nu,p}(\mathbf{R}_+ \times \mathbf{R}_+)$, where $p \geq 1, \nu \in \mathbf{R}$ the Lebesgue space of measurable functions normed by

$$||g(\tau, x)||_{L_{\nu,p}(\mathbf{R}_+ \times \mathbf{R}_+)} = \left(\int_0^\infty \int_0^\infty x^{\nu p - 1} |g(\tau, x)|^p d\tau dx \right)^{1/p}. \tag{2.152}$$

Lemma 2.7. *Let $f(x)$ be from the space $L_{1-\nu,1}(\mathbf{R}_+)$, where $\nu > 0$. Then the operator $KL[f]$ of the index-convolution K-L transform* (2.150) *is bounded from the space $L_{1-\nu,1}(\mathbf{R}_+)$ into the space $L_{\nu,p}(\mathbf{R}_+ \times \mathbf{R}_+)$,$p \geq 1$.*

Proof. Indeed, applying the generalized Minkowski inequality (1.10) and invoking with estimate (1.100) we obtain the chain of relations

$$\begin{aligned}
||KL[f]||_{L_{\nu,p}(\mathbf{R}_+ \times \mathbf{R}_+)} &= \left(\int_0^\infty \int_0^\infty x^{\nu p - 1} \left| \int_0^\infty K_{i\tau}(xy) f(y) dy \right|^p d\tau dx \right)^{1/p} \\
&\leq \int_0^\infty |f(y)| \left(\int_0^\infty \int_0^\infty x^{\nu p - 1} |K_{i\tau}(xy)|^p d\tau dx \right)^{1/p} dy \\
&\leq \int_0^\infty |f(y)| \left(\int_0^\infty \int_0^\infty x^{\nu p - 1} K_0^p(xy \cos\delta) e^{-p\delta\tau} d\tau dx \right)^{1/p} dy, \ \delta \in [0, \pi/2).
\end{aligned} \tag{2.153}$$

Performing the interchange $xy = t$ continue to estimate, namely

$$\begin{aligned}
||KL[f]||_{L_{\nu,p}(\mathbf{R}_+ \times \mathbf{R}_+)} &\leq \int_0^\infty y^{-\nu} |f(y)| dy \left(\int_0^\infty t^{\nu p - 1} K_0^p(t \cos\delta) dt \right)^{1/p} \\
&\times \left(\int_0^\infty e^{-p\delta\tau} d\tau \right)^{1/p} \leq C_{\nu,p,\delta} ||f||_{1-\nu,1},
\end{aligned} \tag{2.154}$$

provided that $\nu > 0, p \geq 1, \delta \in (0, \pi/2)$ and therefore $C_{\nu,p,\delta}$ is a constant due to convergence of the integrals by t and τ (see also asymptotic formulae (1.96)-(1.97)). This completes the proof of Lemma 2.7. •

Drawing a parallel with the results of Section 2.1 (see (2.13)) we define the space of functions $g(\tau, x)$, which can be represented by the index-convolution transform (2.150), where the respective function $f(y)$ belongs to $L_{1-\nu,1}(\mathbf{R}_+)$, namely

$$KL[L_{1-\nu,1}] = \{g : \ g(\tau,x) = KL[f](\tau,x), \ f \in L_{1-\nu,1}(\mathbf{R}_+)\},$$
$$(\tau, x) \in \mathbf{R}_+ \times \mathbf{R}_+. \tag{2.155}$$

To construct the inversion transform of (2.150) we introduce the following operator

$$(I_\varepsilon g)(x) = \frac{2x \sin\varepsilon}{\pi^2} \int_0^\infty \int_0^\infty y \cosh((\pi - \varepsilon)\tau) K_{i\tau}(xy) g(\tau, y) d\tau dy, \tag{2.156}$$

where $\varepsilon \in (0, \pi)$.

Theorem 2.9. *On the functions $g(\tau, x) = KL[f](\tau, x)$, which are represented by the index-convolution Kontorovich-Lebedev transform* (2.150) *with the density $f(y) \in L_{1-\nu,1}(\mathbf{R}_+)$, $0 < \nu < 2$ operator* (2.156) *has the following form*

$$(I_\varepsilon g)(x) = \frac{x \sin\varepsilon}{\pi} \int_0^\infty \frac{f(t)}{x^2 + t^2 - 2xt\cos\varepsilon} dt, \ x > 0. \tag{2.157}$$

Proof. Substituting in formula (2.156) the value of $g(\tau, x)$ estimate the iterated integral for each $x > 0$ and $\varepsilon \in (0, \pi)$, using inequality (1.100) as follows

$$|(I_\varepsilon g)(x)| \leq \frac{2x \sin\varepsilon}{\pi^2} \int_0^\infty \int_0^\infty y \cosh((\pi - \varepsilon)\tau) |K_{i\tau}(xy)|$$
$$\times \int_0^\infty |K_{i\tau}(yt) f(t)| dt d\tau dy$$
$$< C_\varepsilon x \int_0^\infty e^{(\pi - \varepsilon - \delta_1 - \delta_2)\tau} d\tau \int_0^\infty \int_0^\infty y K_0(xy \cos\delta_1) K_0(yt \cos\delta_2) |f(t)| dy dt, \tag{2.158}$$

where $C_\varepsilon > 0$ is a constant and there exist some parameters $\delta_1, \delta_2 \in (0, \pi/2)$ to satisfy the inequality $\pi - \varepsilon - \delta_1 - \delta_2 < 0$ which gives the convergence of the integral by τ. The integral by y can be calculated by formula 2.16.33.1 from Prudnikov et al. [2], namely

$$\int_0^\infty y^{\alpha-1} K_\mu(by) K_\nu(cy) dy$$
$$= 2^{\alpha-3} c^{-\alpha-\mu} \frac{b^\mu}{\Gamma(\alpha)} \Gamma\left(\frac{\alpha+\mu+\nu}{2}\right) \Gamma\left(\frac{\alpha+\mu-\nu}{2}\right) \Gamma\left(\frac{\alpha-\mu+\nu}{2}\right) \Gamma\left(\frac{\alpha-\mu-\nu}{2}\right)$$
$$\times {}_2F_1\left(\frac{\alpha+\mu+\nu}{2}, \frac{\alpha-\mu+\nu}{2}; \alpha; 1 - \frac{b^2}{c^2}\right),$$
$$b + c > 0, \ \Re\alpha > |\Re\mu| + |\Re\nu|. \tag{2.159}$$

Letting here $\alpha = 2, \ \mu = \nu = 0, \ b = x\cos\delta_1, \ c = t\cos\delta_2$ we have the formula

$$\int_0^\infty y K_0(xy\cos\delta_1) K_0(yt\cos\delta_2) dy$$

$$= \frac{1}{2t^2\cos^2\delta_2}{}_2F_1\left(1,\ 1;\ 2; 1-\left(\frac{x\cos\delta_1}{t\cos\delta_2}\right)^2\right). \tag{2.160}$$

Hence for our purposes we may complete the asymptotic formula (1.86) for the Gauss hypergeometric function ${}_2F_1(a,b;c;z)$ in the logarithmic case, i.e. when $b-a=m$, $m=0,1,\ldots$. Precisely, for this case we should write

$${}_2F_1(a,b;c;z) = C_1 z^{-a} + C_2 z^{-b}\log z + O(z^{-a-1}) + O(z^{-b-1}\log z), \tag{2.161}$$

where C_1, C_2 are some positive constants. Therefore, when $t \to 0+$ we obtain that for each $x>0$

$$\frac{1}{2t^2\cos^2\delta_2}{}_2F_1\left(1,\ 1;\ 2; 1-\left(\frac{x\cos\delta_1}{t\cos\delta_2}\right)^2\right) = O(\log t). \tag{2.162}$$

Thus, continuing to estimate integral (2.156) attract now attention to the double integral by y and t. Namely, we have

$$\int_0^\infty\int_0^\infty yK_0(xy\cos\delta_1)K_0(yt\cos\delta_2)|f(t)|dydt$$

$$= \left(\int_0^A + \int_A^\infty\right)|f(t)|dt\int_0^\infty yK_0(xy\cos\delta_1)K_0(yt\cos\delta_2)dy = I_1 + I_2, \tag{2.163}$$

where $A>0$ is some fixed number. Hence for the integral I_1 use the asymptotic value (2.161) and we have

$$I_1 < C\int_0^A |f(t)\log t|dt$$

$$< C_1\int_0^A t^{-\nu}|f(t)|dt < C_1||f||_{1-\nu,1} < \infty. \tag{2.164}$$

The second one can be treated like estimate (2.154) in Lemma 2.7, precisely

$$I_2 < \int_A^\infty t^{-\nu}|f(t)|dt\left(\int_0^\infty y^{(2-\nu)p-1}K_0^p(xy\cos\delta_1)dy\right)^{1/p}$$

$$\times\left(\int_0^\infty y^{\nu q-1}K_0^q(y\cos\delta_2)dy\right)^{1/q} < \infty \tag{2.165}$$

under condition $0<\nu<2$. The above estimates motivate the interchange of integration orders in (2.156) after substitution of the value $g(\tau,x)$ by Fubini's theorem. The inner integral by τ is calculated in 2.16.52.6 of Prudnikov et al. [2], precisely

$$\int_0^\infty \cosh((\pi-\varepsilon)\tau)K_{i\tau}(xy)K_{i\tau}(yt)d\tau$$

$$= \frac{\pi}{2}K_0\left(y\sqrt{x^2+t^2-2xt\cos\varepsilon}\right). \tag{2.166}$$

Appealing to formula (2.140) calculate the integral by y. We find

$$\int_0^\infty yK_0\left(y\sqrt{x^2+t^2-2xt\cos\varepsilon}\right)dy$$

$$= \frac{1}{x^2 + t^2 - 2xt\cos\varepsilon}, \tag{2.167}$$

and arrive finally to the equality (2.157). This completes the proof of Theorem 2.9. •

Now it is not difficult to obtain the inversion theorem for the index-convolution Kontorovich-Lebedev transform (2.150), using approximation properties of the Poisson kernel (1.14).

Theorem 2.10. *Let $g(\tau, x) = KL[f](\tau, x)$ and $f(x) \in L_{1-\nu,1}(\mathbf{R}_+)$, $1 \le \nu < 2$. Then $f(x) = (Ig)(x)$, where $(Ig)(x)$ is meant as the limit*

$$(Ig)(x) = \mathrm{l.i.m.}_{\varepsilon \to 0+}(I_\varepsilon g)(x), \ x > 0 \tag{2.168}$$

by the norm in $L_{1-\nu,1}(\mathbf{R}_+)$. Besides this limit exists also almost everywhere on $\mathbf{R}_+$.

Proof. The proof of this theorem shall follow without difficulties after respective treatment of integral (2.157). Actually, after the interchange $t = x(\cos\varepsilon + v\sin\varepsilon)$, equality (2.157) becomes

$$(I_\varepsilon g)(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{f(x(\cos\varepsilon + v\sin\varepsilon))}{v^2+1} H(v + \cot\varepsilon)dv, \tag{2.169}$$

where $H(x)$ is the Heaviside function. Let us estimate the $L_{1-\nu,1}$-norm of the difference $(I_\varepsilon g) - f$. Indeed, using the generalized Minkowski inequality (1.10) we have

$$||(I_\varepsilon g)||_{1-\nu,1} = \frac{1}{\pi}\left|\left|\int_{-\cot\varepsilon}^{\infty} \frac{f(x(\cos\varepsilon + v\sin\varepsilon))}{v^2+1} dv\right|\right|_{1-\nu,1}$$

$$\le \frac{1}{\pi}\int_{-\cot\varepsilon}^{\infty} \frac{1}{v^2+1}||f(x(\cos\varepsilon + v\sin\varepsilon))||_{1-\nu,1}dv$$

$$= \frac{||f||_{1-\nu,1}}{\pi}\int_{-\cot\varepsilon}^{\infty} \frac{(\cos\varepsilon + v\sin\varepsilon)^{\nu-1}}{v^2+1}dv < \frac{||f||_{1-\nu,1}}{\pi}\int_{-\infty}^{\infty} \frac{(1+|v|)^{\nu-1}}{v^2+1}dv$$

$$= C||f||_{1-\nu,1}, \ 1 \le \nu < 2. \tag{2.170}$$

Now it is clear, that

$$||(I_\varepsilon g) - f||_{1-\nu,1} = \frac{1}{\pi}\left|\left|\int_{-\infty}^{\infty} \frac{1}{v^2+1}(f(x(\cos\varepsilon + v\sin\varepsilon))\right.\right.$$

$$\times\ H(v + \cot\varepsilon) - f(x))\,dv||_{1-\nu,1}$$

$$\le \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{v^2+1}||f(x(\cos\varepsilon + v\sin\varepsilon))$$

$$\times H(v + \cot\varepsilon) - f(x)||_{1-\nu,1}dv \tag{2.171}$$

and it tends to zero, when $\varepsilon \to 0+$ due to the Lebesgue theorem. To establish the limit almost everywhere on $\mathbf{R}_+$ it is enough to call Theorem 1.4. Theorem 2.10 is proved. •

Thus in view of the previous theorem we obtain the uniform estimate of kind

$$||(I_\varepsilon g)||_{1-\nu} \le ||(Ig)||_{1-\nu}, \ g \in KL[L_{1-\nu,1}]. \tag{2.172}$$

It follows also that $KL[f](\tau, x) \equiv 0, \ f(t) \in L_{1-\nu,1}(\mathbf{R}_+)$ if and only if $f(t) = 0$ almost everywhere on $\mathbf{R}_+$. So in the space $KL[L_{1-\nu,1}]$ we introduce norm by the equality

$$||g||_{KL[L_{1-\nu,1}]} = ||f||_{1-\nu}, \quad g = KL[f]. \tag{2.173}$$

Obviously, the space $KL[L_{1-\nu,1}]$ is a Banach one with norm (2.173) and isometric $L_{1-\nu,1}(\mathbf{R}_+)$. Apart from Theorem 2.10 let us give some sufficient conditions of belonging of an arbitrary function $g(\tau, x)$ defined on $\mathbf{R}_+ \times \mathbf{R}_+$ to the space $KL[L_{1-\nu,1}]$. For this start from the evaluation of the composition

$$g_\varepsilon(\tau, x) \equiv \int_0^\infty K_{i\tau}(xy)(I_\varepsilon g)(y)dy. \tag{2.174}$$

At least for the "good" functions (the infinitely smooth ones with the compact support) we easily change the order of integration and invoking with (2.156) obtain

$$g_\varepsilon(\tau, x) = \frac{2\sin\varepsilon}{\pi^2} \int_0^\infty \int_0^\infty t\cosh((\pi - \varepsilon)\beta) g(\beta, t) d\beta dt$$

$$\times \int_0^\infty y K_{i\tau}(xy) K_{i\beta}(ty) dy. \tag{2.175}$$

Operator (2.174) is bounded one from the space $L_{1-\nu,1}(\mathbf{R}_+)$, $1 \le \nu < 2$ into the space $L_{\nu,p}(\mathbf{R}_+ \times \mathbf{R}_+)$, $p \ge 1$ under Theorem 2.10. Let us estimate the left-hand side of equality (2.175) using the generalized Minkowski inequality (1.10). We have

$$||g_\varepsilon(\tau, x)||_{L_{\nu,p}(\mathbf{R}_+\times\mathbf{R}_+)} = \frac{2\sin\varepsilon}{\pi^2} \left(\int_0^\infty \int_0^\infty x^{\nu p - 1} \right.$$

$$\times \left| \int_0^\infty \int_0^\infty t\cosh((\pi - \varepsilon)\beta) g(\beta, t) d\beta dt \right.$$

$$\left. \left. \times \int_0^\infty y K_{i\tau}(xy) K_{i\beta}(ty) dy \right|^p d\tau dx \right)^{1/p}$$

$$\le \frac{2\sin\varepsilon}{\pi^2} \int_0^\infty \int_0^\infty t\cosh((\pi - \varepsilon)\beta) |g(\beta, t)| d\beta dt$$

$$\times \int_0^\infty y |K_{i\beta}(ty)| dy$$

$$\times \left(\int_0^\infty \int_0^\infty x^{\nu p - 1} |K_{i\tau}(xy)|^p dx d\tau \right)^{1/p}. \tag{2.176}$$

Making use inequality (1.100) which implies that there exist parameters $\delta_1, \delta_2 \in (0, \pi/2)$ such that with simple interchanges the previous chain of inequalities leads to

$$||g_\varepsilon(\tau, x)||_{L_{\nu,p}(\mathbf{R}_+\times\mathbf{R}_+)} \le \frac{2\sin\varepsilon}{\pi^2} \int_0^\infty \int_0^\infty t^{\nu-1} e^{(\pi-\varepsilon-\delta_1)\beta} |g(\beta, t)| d\beta dt$$

$$\times \int_0^\infty y^{1-\nu} K_0(y\cos\delta_1)dy$$

$$\times \left(\int_0^\infty e^{-p\delta_2\tau}d\tau\right)^{1/p} \left(\int_0^\infty x^{\nu p-1} K_0^p(x\cos\delta_2)dx\right)^{1/p} < \infty, \tag{2.177}$$

if ν satisfies the condition $0 < \nu < 2$ and

$$g(\tau, x) \in L_{\nu,1}\left(\mathbf{R}_+ \times \mathbf{R}_+; e^{(\pi-\varepsilon-\delta_1)\beta}\right). \tag{2.178}$$

Thus appealing to formula (2.159) one can write composition (2.174) as

$$g_\varepsilon(\tau, x) = \frac{\sin\varepsilon}{4}\int_0^\infty \int_0^\infty \cosh((\pi-\varepsilon)\beta) g(\beta, t) d\beta \frac{dt}{t}$$

$$\times \left(\frac{x}{t}\right)^{i\tau} \frac{\tau^2 - \beta^2}{\sinh(\pi(\tau+\beta)/2)\sinh(\pi(\tau-\beta)/2)}$$

$$\times {}_2F_1\left(1 + \frac{i(\tau+\beta)}{2},\ 1 + \frac{i(\beta-\tau)}{2};\ 2; 1 - \frac{x^2}{t^2}\right). \tag{2.179}$$

Passing through to the limit in mean sense in equality (2.174) under condition (2.168) we obtain the desired result. So finally we proved the theorem that contains sufficient conditions of belonging an arbitrary function $g(\tau, x)$ to the range of the index-convolution Kontorovich-Lebedev transform (2.150).

Theorem 2.11. *An arbitrary function $g(\tau, x)$ belongs to the space $KL[L_{1-\nu,1}]$, $0 < \nu < 2$ under following conditions*

$$g(\tau, x) = \text{l.i.m.}_{\varepsilon\to 0+} g_\varepsilon(\tau, x), \tag{2.180}$$

where $g_\varepsilon(\tau, x)$ defined by formula (2.179) and the limit in (2.180) is meant by the norm of the space $L_{\nu,p}(\mathbf{R}_+ \times \mathbf{R}_+)$. In addition, $g(\tau, x)$ satisfies to condition (2.178) and $(I_\varepsilon g)(x)$ converges by the norm of the space $L_{1-\nu,1}(\mathbf{R}_+)$.

At the end of this chapter we give an example of the index-convolution transform (2.150) and the limit relation (2.168). Note that to derive a wide set of such examples we need to extend the range of the parameter ν in Theorem 2.10. This we can do returning, for instance to estimate (2.170). Indeed, calculate the integral

$$I_\varepsilon^\nu = \int_{-\cot\varepsilon}^\infty \frac{(\cos\varepsilon + v\sin\varepsilon)^{\nu-1}}{v^2+1}dv = \sin^{\nu-1}\varepsilon \int_0^\infty \frac{y^{\nu-1}}{y^2 - 2y\cot\varepsilon + \cot^2\varepsilon + 1}dy \tag{2.181}$$

using formula 2.2.9.7 from Prudnikov et al. [1] under conditions $0 < \nu < 2$, $\varepsilon \in (0, \pi)$. As result invoking with relation 7.3.1.91 from Prudnikov et al. [3] we obtain

$$I_\varepsilon^\nu = \sin\varepsilon\Gamma(\nu)\Gamma(2-\nu) {}_2F_1\left(\frac{\nu}{2},\ 1 - \frac{\nu}{2};\ \frac{3}{2};\ \sin^2\varepsilon\right)$$

$$= \Gamma(\nu)\Gamma(2-\nu)\frac{\sin((\nu-1)\varepsilon)}{\nu-1}. \tag{2.182}$$

Hence it follows that $|I_\varepsilon| < C$, $\varepsilon \in (0,\pi)$, $0 < \nu < 2$. Take now as the example $f(x) = e^{-x}/\sqrt{x}$. Obviously, $f(x) \in L_{1-\nu,1}(\mathbf{R}_+)$ when $\nu < 1/2$, because the integral

$$\int_0^\infty e^{-y} y^{-\nu-1/2} dy = \Gamma\left(\frac{1}{2} - \nu\right) < \infty, \ \nu < 1/2.$$

According to formula 2.16.6.3 from Prudnikov et al. [2] (see also (1.102)) the respective integral transform (2.150) becomes as

$$g(\tau, x) = \int_0^\infty K_{i\tau}(xy) e^{-y} \frac{dy}{\sqrt{y}}$$

$$= \sqrt{\frac{\pi}{2x}} \frac{\pi}{\cosh(\pi\tau/2)} P_{-1/2+i\tau}\left(\frac{1}{x}\right), \tag{2.183}$$

where $P_{-1/2+i\tau}(1/x)$ is particular case of the associated Legendre function (1.56). Therefore, substituting this result into (2.156) find that for all $x > 0$ the following limit equality is true

$$e^{-x} = x^{3/2} \sqrt{\frac{2}{\pi}} \lim_{\varepsilon \to 0+} \sin\varepsilon \int_0^\infty \int_0^\infty \frac{\cosh((\pi - \varepsilon)\tau)}{\cosh(\pi\tau/2)}$$

$$\times \sqrt{y} K_{i\tau}(xy) P_{-1/2+i\tau}\left(\frac{1}{y}\right) d\tau dy. \tag{2.184}$$

Chapter 3

The Mehler-Fock Transform

This chapter deals with one of the famous index transforms by the index of the associated Legendre function of the first kind (1.55) introduced by Mehler [1] and Fock [1]. The respective expansion of an arbitrary function is given by formula (1.233), when we put $\mu = 0$, $\nu = i\tau - 1/2$ in formula (1.55). This integral transform as the Kontorovich-Lebedev transform being investigated in previous chapter has important applications in mathematical physics although we shall attract our attention to pure mathematical problems as its mapping properties and the inversion in L_p-spaces. Note that the detailed investigation of the Mehler-Fock integral transform in L_1 and L_2 spaces was given by Lebedev [4], [6]-[7], [9] and recently by the author in the monograph by Yakubovich and Luchko [2]. As usually let us list other papers from the bibliography which are devoted to the Mehler-Fock transform and its applications. We note Belova [1], Brychkov et al. [1], Buggle [1], Ditkin and Prudnikov [1], Glaeske [3], [4], Hayek et al. (1990) [1], Hayek and Gonzalez [1], Lebedev and Skalskaya [1]-[2], [4], Lowndes [4]-[5], Mandal [1]-[2], Mandal and Roy [1], Mandal, N. and Mandal, B.N. [1], Markushevich [1], Oberhettinger and Higgins [1], Orlyuk [1], Pathak and Pandey [1], Rosenthal [1], Sneddon [2], Srivastava [1]-[2], Stolov [1], Tiwari and Pandey [1], Vilenkin [1], Yakubovich [7].

3.1 Definition. Inversion in L_p-space

We start to discuss here the Mehler-Fock transform of the following type

$$MF[f](\tau) = \frac{\pi}{2}\int_0^\infty P_{-1/2+i\tau/2}(2y^2+1)f(y)dy, \quad \tau \geq 0, \tag{3.1}$$

where $P_{-1/2+i\tau/2}(2x^2+1)$ is the associated Legendre function of the first kind (1.55) of the argument $2x^2+1$ and the index $-1/2+i\tau$. As we already noted above the Mehler-Fock transform is very important as a basic transform among the class of index transforms being related to the Kontorovich-Lebedev transform (2.1). The reader can find interesting composition theorems for index transforms in Yakubovich and

Luchko [2] established by the author that show connection between index transforms and convolution transforms of the Mellin type considered in Chapter 1. The Mehler-Fock transform as well as the Kontorovich- Lebedev transform mentioned above play a key role within these compositions.

To investigate the Mehler-Fock transform in $L_p(\mathbf{R}_+)$ first we need to know some estimates concerning the kernel of integral (3.1). For these purposes we shall use the integral representation

$$\frac{\pi}{2\cosh(\pi\tau/2)}P_{-1/2+i\tau/2}(2x^2+1) = \int_0^\infty J_0(xy)K_{i\tau}(y)dy, \quad \tau, x > 0, \tag{3.2}$$

which can be seen from Prudnikov et al. ([2], formula 2.16.21.1), where $J_\nu(z)$ is the Bessel function (1.88) and $K_\nu(z)$ is the Macdonald function (1.91).

Invoking with representation (1.99) substitute it in formula (3.2) and applying Fubini's theorem in view of estimate (1.100) and asymptotic of the Bessel function at the points zero and infinity we change the order of integration in the obtained iterated integral. As a result calculate the inside integral by formula 2.12.8.3 from Prudnikov et al. [2]. It gives us that

$$\frac{\pi}{\cosh(\pi\tau/2)}P_{-1/2+i\tau/2}(2x^2+1) = \int_{i\delta-\infty}^{i\delta+\infty} \frac{e^{i\tau\beta}}{\sqrt{x^2+\cosh^2\beta}}d\beta, \quad \tau, x > 0, \tag{3.3}$$

where we choose the main value of the square root in the integrand .

Useful estimates of the given Legendre function for further applications are established by the following lemma.

Lemma 3.1.. *The uniform estimates of the Legendre function* (3.3) *by variable* $x > 0$ *and* $\tau \geq 0$ *are true*

$$\frac{1}{\cosh(\pi\tau/2)}|P_{-1/2+i\tau/2}(2x^2+1)|$$

$$\leq \frac{e^{-\delta\tau}}{\sqrt{\cos 2\delta}}P_{-1/2}\left(2\frac{x^2+\sin^2\delta}{\cos 2\delta}+1\right), \ \delta \in \left[0, \frac{\pi}{4}\right), \tag{3.4}$$

$$\frac{1}{\cosh(\pi\tau/2)}|P_{-1/2+i\tau/2}(2x^2+1)|$$

$$\leq \sqrt{\pi}\Gamma(1/4)2^{-1/4}e^{-\pi\tau/4}x^{1/4}(x^2+1)^{1/8}(4x^4+4x^2+1)^{1/8}$$

$$\times P^{1/4}_{-1/2}(8x^4+8x^2+1), \ \delta = \pi/4, \tag{3.5}$$

$$\frac{1}{\cosh(\pi\tau/2)}|P_{-1/2+i\tau/2}(2x^2+1)|$$

$$\leq e^{-\delta\tau}\left[C_1P_{-1/2}(\cosh a) + C_2\frac{\sqrt{\cosh a}}{\sinh a} + C_3\frac{1}{\sqrt{\cosh a}}\right], \ \delta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right), \tag{3.6}$$

where C_i, $i = 1, 2, 3$ *are absolute positive constants and*

$$\cosh a = -\frac{2x^2+1}{\cos 2\delta}. \tag{3.7}$$

Proof. Indeed, to show (3.4) from representation (3.3) we have the relations

$$\frac{\pi}{\cosh(\pi\tau/2)}|P_{-1/2+i\tau/2}(2x^2+1)| \le e^{-\delta\tau}\int_{-\infty}^{\infty}\frac{d\beta}{\sqrt{|x^2+\cosh^2(\beta+i\delta)|}}$$

$$\le e^{-\delta\tau}\int_{-\infty}^{\infty}\frac{d\beta}{\sqrt{|x^2+\sin^2\delta+\cos 2\delta\cosh^2\beta|}}$$

$$= \frac{e^{-\delta\tau}}{\sqrt{\cos 2\delta}}\int_{-\infty}^{\infty}\frac{d\beta}{\sqrt{\dfrac{x^2+\sin^2\delta}{\cos 2\delta}+\cosh^2\beta}}$$

$$= \frac{\pi e^{-\delta\tau}}{\sqrt{\cos 2\delta}}P_{-1/2}\left(2\frac{x^2+\sin^2\delta}{\cos 2\delta}+1\right). \tag{3.8}$$

This proves inequality (3.4). In second case, we find

$$\frac{\pi}{\cosh(\pi\tau/2)}|P_{-1/2+i\tau/2}(2x^2+1)| \le e^{-\pi\tau/4}\int_{-\infty}^{\infty}\frac{d\beta}{\sqrt{|x^2+\cosh^2(\beta+i\pi/4)|}}$$

$$= e^{-\pi\tau/4}2^{-1/4}\int_0^{\infty}\frac{d\beta}{(8x^4+8x^2+1+\cosh\beta)^{1/4}}$$

$$= \sqrt{\pi}\Gamma(1/4)2^{-1/4}e^{-\pi\tau/4}x^{1/4}(x^2+1)^{1/8}(4x^4+4x^2+1)^{1/8}$$

$$\times P_{-1/2}^{1/4}(8x^4+8x^2+1) \tag{3.9}$$

according to integral 2.4.6.9 from Prudnikov et al. [1]. For third case we have

$$\frac{\pi}{\cosh(\pi\tau/2)}|P_{-1/2+i\tau/2}(2x^2+1)| \le e^{-\delta\tau}\int_{-\infty}^{\infty}\frac{d\beta}{\sqrt{|x^2+\cosh^2(\beta+i\delta)|}}$$

$$\le \frac{\sqrt{2}e^{-\delta\tau}}{\sqrt{|\cos 2\delta|}}\left(\int_0^a\frac{d\beta}{\sqrt{\cosh a-\cosh\beta}}+\int_a^{\infty}\frac{d\beta}{\sqrt{\cosh\beta-\cosh a}}\right)$$

$$= I(\tau, x), \tag{3.10}$$

where $\cosh a$ is defined by formula (3.7). Hence using formula 2.4.6.1 from Prudnikov et al. [1] and choosing a constant $B > 1$, we obtain

$$I(\tau, x) = \frac{\pi e^{-\delta\tau}}{\sqrt{|\cos 2\delta|}}P_{-1/2}(\cosh a)$$

$$+\frac{\sqrt{2}e^{-\delta\tau}}{\sqrt{|\cos 2\delta|}}\left(\sqrt{\cosh a}\int_1^B \frac{du}{\sqrt{u-1}\sqrt{u^2\cosh^2 a-1}}\right.$$

$$\left.+\frac{1}{\sqrt{\cosh a}}\int_B^\infty \frac{du}{\sqrt{u-1}\sqrt{u^2-1/\cosh^2 a}}\right)$$

$$\leq \frac{e^{-\delta\tau}}{\sqrt{|\cos 2\delta|}}\left[\pi P_{-1/2}(\cosh a)+C_4\frac{\sqrt{\cosh a}}{\sinh a}+C_5\frac{1}{\sqrt{\cosh a}}\right], \tag{3.11}$$

where C_4, C_5 are some positive constants. This leads us to estimate (3.6) which completes the proof of Lemma 3.1.•

On the other hand, invoking with representation (1.55), asymptotic formula (1.86) and value (3.5) from the asymptotic behavior of the Gauss hypergeometric function (1.47) we conclude that

$$P_{-1/2}(\cosh a) = O(1), \; a \to 0, \tag{3.12}$$

$$P_{-1/2}(\cosh a) = O(P_{-1/2}(2x^2+1)) = O\left(\frac{1}{x}\right), \; x \to \infty, \tag{3.13}$$

$$P_{-1/2}\left(2\frac{x^2+\sin^2\delta}{\cos 2\delta}+1\right) = O(P_{-1/2}(2x^2+1)) = O\left(\frac{1}{x}\right), \; x \to \infty, \tag{3.14}$$

$$x^{1/4}(x^2+1)^{1/8}(4x^4+4x^2+1)^{1/8}P^{1/4}_{-1/2}(8x^4+8x^2+1)$$

$$= O(P_{-1/2}(2x^2+1)) = O\left(\frac{1}{x}\right), \; x \to \infty. \tag{3.15}$$

So we find from estimates (3.4)-(3.6) that

$$|P_{-1/2+i\tau/2}(2x^2+1)| \leq Ce^{(\pi/2-\delta)\tau}P_{-1/2}(2x^2+1) \tag{3.16}$$

for $\delta \in [0, \pi/2)$, $\tau \geq 0$ and $x > 0$.

Let us consider the Mehler-Fock transform (3.1) when the density $f(x)$ belongs to the space $L_p(\mathbf{R}_+)$, $1 \leq p < \infty$ with norm (1.1). As is evident from the Hölder inequality (1.8) and from the asymptotic behavior of the Legendre function (3.12)-(3.15) integral (3.1) converges absolutely for any $p \geq 1$. The corresponding space of functions $g(\tau)$, represented by the Mehler-Fock transform (3.1) of functions belonging to $L_p(\mathbf{R}_+)$ denoted by

$$MF(L_p) = \{g : g(\tau) = MF[f](\tau) : f \in L_p(\mathbf{R}_+)\}, \; p \geq 1. \tag{3.17}$$

As in the case of the Kontorovich-Lebedev transform we investigate now the mapping properties of the Mehler-Fock transform and show that the operator $MF[f]$ is a bounded mapping from $L_p(\mathbf{R}_+)$, $1 \leq p < \infty$ into $L_r(\mathbf{R}_+; e^{-\alpha\tau})$, $1 \leq r \leq \infty$ with exponential weight $e^{-\alpha\tau}$, where $\alpha > 0$ and p and r have no dependence.

Actually, making use of the generalized Minkowski inequality (1.10) and estimate (3.16) we have

$$||MF[f]||_{L_r(\mathbf{R}_+;e^{-\alpha\tau})} \leq \frac{\pi}{2}\int_0^\infty |f(y)| \left(\int_0^\infty e^{-\alpha\tau}|P_{-1/2+i\tau/2}(2y^2+1)|^r d\tau\right)^{1/r} dy$$

$$\leq C\int_0^\infty |f(y)|P_{-1/2}(2y^2+1)dy \left(\int_0^\infty e^{(\pi/2-\alpha/r-\delta)r\tau}d\tau\right)^{1/r}$$

$$\leq C_\delta\left(\int_0^1 |f(y)|dy + \int_1^\infty |f(y)|\frac{dy}{y}\right)$$

$$\leq C_{\delta 1}||f||_{L_p},\ 1 \leq p < \infty, \tag{3.18}$$

where $C_\delta, C_{\delta 1}$ are positive constants, $\pi/2-\alpha/r < \delta < \pi/2$. Here, in the last inequality we applied additionally the Hölder inequality (1.8).

In order to describe the introduced space $MF(L_p)$ (3.17), let us consider the following operator

$$(I_\varepsilon g)(x) = \frac{x^{1-\varepsilon}}{\pi}\int_0^\infty \frac{\tau\sinh((\pi-\varepsilon)\tau)}{\cosh^2(\pi\tau/2)}P_{-1/2+i\tau/2}(2x^2+1)g(\tau)d\tau, \quad x>0, \tag{3.19}$$

where $\varepsilon \in (0,1)$.

Theorem 3.1. *For the Mehler-Fock transform $g(\tau) = MF[f](\tau)$ of $f(y) \in L_p(\mathbf{R}_+), 1 \leq p < \infty$ operator* (3.19) *has the form*

$$(I_\varepsilon g)(x) = \int_0^\infty I(x,y,\varepsilon)f(y)dy, \quad x>0, \tag{3.20}$$

where

$$I(x,y,\varepsilon) = \frac{2x^{1-\varepsilon}\sin\varepsilon}{\pi}$$

$$\times\int_0^\infty \frac{u(x^2+y^2u^2+1+u^2-2u\cos\varepsilon)}{\left[(x^2+y^2u^2+1+u^2-2u\cos\varepsilon)^2-4x^2y^2u^2\right]^{3/2}}du. \tag{3.21}$$

Proof. Substituting the value of $g(\tau)$ as the Mehler-Fock transform (3.1) we obtain the following iterated integral

$$(I_\varepsilon g)(x) = \frac{x^{1-\varepsilon}}{2}\int_0^\infty \frac{\tau\sinh((\pi-\varepsilon)\tau)}{\cosh^2(\pi\tau/2)}P_{-1/2+i\tau/2}(2x^2+1)$$

$$\times\int_0^\infty P_{-1/2+i\tau/2}(2y^2+1)f(y)dyd\tau, \tag{3.22}$$

which is absolutely convergent for any $f(x) \in L_p(\mathbf{R}_+)$, $1 \leq p < \infty$ by using the estimate (3.16). Now we treat the inner integral by index of the Legendre functions

$$I(x,y,\varepsilon) = \frac{x^{1-\varepsilon}}{2}\int_0^\infty \frac{\tau\sinh((\pi-\varepsilon)\tau)}{\cosh^2(\pi\tau/2)}P_{-1/2+i\tau/2}(2x^2+1)P_{-1/2+i\tau/2}(2y^2+1)d\tau \tag{3.23}$$

and we shall prove that it coincides with (3.21). Invoking with representation (3.2), we have

$$I(x,y,\varepsilon)=\frac{2x^{1-\varepsilon}}{\pi^2}\int_0^\infty \tau\sinh((\pi-\varepsilon)\tau)d\tau\int_0^\infty J_0(xv)K_{i\tau}(v)dv$$

$$\times\int_0^\infty J_0(yu)K_{i\tau}(u)dud\tau. \tag{3.24}$$

Thus the key problem comes to change the order of integration in (3.24) and apply the Fubini theorem. From the estimate of the Macdonald function (1.100) and uniform boundedness of the Bessel function $J_0(xy)$ for positive variables x, y we have the inequality

$$|I(x,y,\varepsilon)|\leq Cx^{1-\varepsilon}\left[\int_0^\infty \tau e^{(\pi-\varepsilon-2\delta)\tau}d\tau\right]$$

$$\times\int_0^\infty K_0(v\cos\delta)dv\int_0^\infty K_0(u\cos\delta)du<+\infty, \tag{3.25}$$

according to asymptotic behavior of the Macdonald function (1.96)-(1.97), and we choose the parameter δ in (3.25) as $(\pi-\varepsilon)/2<\delta<\pi/2$. Now first calculate the inner integral by τ

$$I_1(u,v,\varepsilon)=\frac{2}{\pi^2}\int_0^\infty \tau\sinh((\pi-\varepsilon)\tau)K_{i\tau}(v)K_{i\tau}(u)d\tau, \tag{3.26}$$

using formula 2.16.52.6 from Prudnikov et al. [2] (see also (2.166)). We have

$$\int_0^\infty \cosh((\pi-\varepsilon)\tau)K_{i\tau}(v)K_{i\tau}(u)d\tau=\frac{\pi}{2}K_0(\sqrt{u^2+v^2-2uv\cos\varepsilon}). \tag{3.27}$$

Then by differentiating the integral (3.27) by parameter ε find (see also Erdélyi et al. [1]) that

$$I_1(u,v,\varepsilon)=-\frac{1}{\pi}\frac{\partial}{\partial\varepsilon}K_0(\sqrt{u^2+v^2-2uv\cos\varepsilon}). \tag{3.28}$$

The substitution of (3.28) into (3.24) yields the double integral

$$I(x,y,\varepsilon)=-\frac{x^{1-\varepsilon}}{\pi}\frac{\partial}{\partial\varepsilon}\int_0^\infty\int_0^\infty J_0(xv)J_0(yu)K_0(\sqrt{u^2+v^2-2uv\cos\varepsilon})dudv, \tag{3.29}$$

where, apparently, we need to justify the validity of differentiability by ε under the sign of the double integral. For this performing the differentiation of (3.28), we arrive

$$I_1(u,v,\varepsilon)=\frac{\sin\varepsilon}{\pi}\frac{uvK_1(\sqrt{u^2+v^2-2uv\cos\varepsilon})}{\sqrt{u^2+v^2-2uv\cos\varepsilon}}, \tag{3.30}$$

and by the polar coordinates $v=r\cos\varphi,\ u=r\sin\varphi, r>0,\ 0\leq\varphi\leq\pi/2$ the integral (3.29) can be written in the form

$$I(x,y,\varepsilon)=\frac{x^{1-\varepsilon}\sin\varepsilon}{2\pi}\int_0^{\pi/2}\frac{\sin 2\varphi}{\sqrt{1-\sin 2\varphi\cos\varepsilon}}d\varphi$$

$$\times \int_0^\infty r^2 J_0(xr\cos\varphi)J_0(yr\sin\varphi)K_1(r\sqrt{1-\sin 2\varphi\cos\varepsilon})dr. \tag{3.31}$$

Hence its uniform convergence by $\varepsilon \geq \varepsilon_0 > 0$ follows from the estimate

$$|I(x,y,\varepsilon)| < Cy^{-1/2}x^{1/2-\varepsilon}\sin\varepsilon \int_0^{\pi/2} \frac{d\varphi}{(1-\sin 2\varphi\cos\varepsilon)^{3/2}} \int_0^\infty tK_1(t)dt$$

$$< C_1 \frac{y^{-1/2}x^{1/2-\varepsilon}}{\sin\varepsilon} \int_{-\infty}^\infty \frac{\sqrt{1+(1+|t|)^2}}{(t^2+1)^{3/2}} dt$$

$$< C_2 \frac{y^{-1/2}x^{1/2-\varepsilon}}{\varepsilon},\ 0 < \varepsilon < 1, \tag{3.32}$$

which can be deduced by using the inequality

$$|J_\nu(x)| \leq \frac{C}{\sqrt{x}},\ x > 0, \Re\nu \geq -\frac{1}{2}, \tag{3.33}$$

by changing the variable $\tan\varphi = t\sin\varepsilon + \cos\varepsilon$, and by the asymptotic of the Macdonald function $K_1(t)$ (1.96)-(1.97) that provides the convergence of the respective integral in (3.32). Thus we obtain the following representation

$$I(x,y,\varepsilon) = -\frac{x^{1-\varepsilon}}{\pi}\frac{\partial}{\partial\varepsilon}\int_0^{\pi/2} d\varphi \int_0^\infty rJ_0(xr\cos\varphi)J_0(yr\sin\varphi)$$

$$\times K_0(r\sqrt{1-\sin 2\varphi\cos\varepsilon})dr. \tag{3.34}$$

However the integral by r is evaluated by formula 2.16.37.2 from Prudnikov et al. [2], and representation (3.34) takes the form

$$I(x,y,\varepsilon) = -\frac{x^{1-\varepsilon}}{\pi}\frac{\partial}{\partial\varepsilon}\int_0^{\pi/2} \left[(x\cos\varphi - y\sin\varphi)^2 + 1 - \sin 2\varphi\cos\varepsilon\right]^{-1/2}$$

$$\times \left[(x\cos\varphi + y\sin\varphi)^2 + 1 - \sin 2\varphi\cos\varepsilon\right]^{-1/2} d\varphi. \tag{3.35}$$

Let us change the variable $\tan\varphi = u$ in (3.35) and simple transformations carry out the differentiation by ε. Hence we immediately obtain formula (3.20). Theorem 3.1 is proved. •

The inversion formula for the Mehler-Fock transform (3.1) shall be established by the following

Theorem 3.2. *Let $g(\tau) = MF[f](\tau)$ for $f(y) \in L_p(\mathbf{R}_+), 1 < p < \infty$. Then*

$$f(x) = (Ig)(x), \tag{3.36}$$

where

$$(Ig)(x) = \text{l.i.m.}_{\varepsilon\to 0+}(I_\varepsilon g)(x),\ x > 0 \tag{3.37}$$

and $(I_\varepsilon g)(x)$ is defined in (3.19). Here the limit is meant in the norm of $L_p(\mathbf{R}_+)$. Moreover, the limit in (3.37) exists almost everywhere on $\mathbf{R}_+$.

Proof. Inequality (3.16) implies the uniform estimate for the function $I(x,y,\varepsilon)$ in (3.23) by $x, y > 0$ and $\varepsilon \in (0,1)$, namely

$$|I(x,y,\varepsilon)| < Cx^{1-\varepsilon}P_{-1/2}(2x^2+1)P_{-1/2}(2y^2+1). \tag{3.38}$$

By the replacement $y = x(1+t\varepsilon)$ in integral (3.20) we obtain

$$(I_\varepsilon g)(x) = x\varepsilon \int_{-1/\varepsilon}^{\infty} I(x, x(1+t\varepsilon), \varepsilon) f(x(1+t\varepsilon))dt. \tag{3.39}$$

Now we may estimate more precisely the right-hand side of inequality (3.38). For this use the representation of the Legendre function $P_{-1/2}(2x^2+1)$ through the Meijer G-function (1.59). Indeed, let us consider formula (1.122). Letting there $\mu = 0$, $\nu = -1/2$ and substituting instead of x the variable x^2 we obtain the following Mellin-Barnes integral

$$P_{-1/2}(2x^2+1) = \frac{1}{2\pi^2 i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma(s)\Gamma(1/2-s)\Gamma(1/2-s)}{\Gamma(1-s)} x^{-2s} ds, \tag{3.40}$$

where according to the theory of Meijer's G-function we chose the contour at respective formula (1.59) as a vertical straight line at the complex plane s with $\Re s = \gamma$, $0 < \gamma < 1/2$ to separate the series of left and right poles of integrand (3.40). As a result we find obvious estimate as

$$P_{-1/2}(2x^2+1) < Cx^{-2\gamma} \tag{3.41}$$

uniformly for all $x > 0$, because integral (3.40) is absolutely convergent due to the corollary (1.33) of Stirling's formula as the absolute value of the argument diverges.

With the aid of the generalized Minkowski inequality (1.10) we estimate now the $L_p(\mathbf{R}_+)$-norm of operator (3.39), namely

$$\begin{aligned} ||\,(I_\varepsilon g)\,||_{L_p} &\le \int_{-1/\varepsilon}^{\infty} ||f(x(1+t\varepsilon))x\varepsilon I(x, x(1+t\varepsilon),\varepsilon)||_{L_p} dt \\ &= \int_{-1/\varepsilon}^{0} ||f(x(1+t\varepsilon))x\varepsilon I(x, x(1+t\varepsilon),\varepsilon)||_{L_p} dt \\ &+ \int_{0}^{\infty} ||f(x(1+t\varepsilon))x\varepsilon I(x, x(1+t\varepsilon),\varepsilon)||_{L_p} dt \\ &= I_1 + I_2. \end{aligned} \tag{3.42}$$

To estimate I_1 and I_2 use inequalities (3.38), (3.41). Indeed, we have

$$\begin{aligned} |xI(x, x(1+t\varepsilon),\varepsilon)| &< Cx^{2-\varepsilon}P_{-1/2}(2x^2+1)P_{-1/2}(2x^2(1+t\varepsilon)^2+1) \\ &< C_2 x^{2(1-\gamma_1-\gamma_2)-\varepsilon}(1+t\varepsilon)^{-2\gamma_2}, \end{aligned} \tag{3.43}$$

where there exist such parameters γ_1, γ_2 for the Mellin-Barnes integral (3.40) that $0 < \gamma_1,\ \gamma_2 < 1/2$ and $1 - \varepsilon/2 < \gamma_1 + \gamma_2$. Hence I_1 can be estimated as

$$I_1 \leq \int_{-1/\varepsilon}^{0} ||f(x(1+t\varepsilon))x\varepsilon I(x, x(1+t\varepsilon), \varepsilon)||_{L_p(0,1)} dt$$

$$+ \int_{-1/\varepsilon}^{0} ||f(x(1+t\varepsilon))x\varepsilon I(x, x(1+t\varepsilon), \varepsilon)||_{L_p(1,\infty)} dt$$

$$\leq A_1 ||f(x)x^{1-2\gamma_1-\varepsilon}||_{L_p(0,1)} \int_{-1/\varepsilon}^{0} (1+t\varepsilon)^{2\gamma_1+\varepsilon-1/p-1} dt$$

$$+A_2||f||_{L_p(1,\infty)} \int_{-1/\varepsilon}^{0} (1+t\varepsilon)^{-2\gamma_2-1/p} dt < A||f||_{L_p(\mathbf{R}_+)}, \quad A_1, A_2, A > 0, \tag{3.44}$$

if we choose $\gamma_1,\ \gamma_2$, further, as $2\gamma_1 > 1/p - \varepsilon,\ \gamma_2 < 1 - 1/p$. Similarly, we have by changing the variable $1 + \varepsilon t = u$

$$I_2 \leq B_1 ||f(x)x^{2-2\gamma_1-\varepsilon}||_{L_p(0,1)} \int_{1}^{\infty} u^{2\gamma_1+\varepsilon-1/p-2} du$$

$$+B_2||f(x)x^{2(1-\gamma_1-\gamma_2)-\varepsilon}||_{L_p(1,\infty)} \int_{1}^{\infty} u^{2\gamma_1+\varepsilon-1/p-2} du < B||f||_{L_p(\mathbf{R}_+)}, \quad B_1, B_2, B > 0. \tag{3.45}$$

Finally from the above estimates (3.44)-(3.45) we obtain the inequality for L_p-norm estimation of operator (3.19)

$$||\,(I_\varepsilon g)\,||_{L_p} < C||f||_{L_p}, \tag{3.46}$$

where C is an absolute positive constant.

Let us now proceed to estimate the norm of difference $||(I_\varepsilon g) - f||_{L_p}$ and to show that it tends to zero when $\varepsilon \to 0+$. We first prove the relation

$$\lim_{\varepsilon \to 0+} \varepsilon I(x, x(1+t\varepsilon), \varepsilon) = \frac{1}{\pi} \frac{\sqrt{x^2+1}}{x^2t^2 + x^2 + 1}, x > 0, t \in \mathbf{R}, \tag{3.47}$$

by virtue of representation (3.21). Indeed, substituting $y = x(1+t\varepsilon)$ and change the variable $u = 1 + v\varepsilon$ we obtain that

$$\varepsilon I(x, x(1+t\varepsilon), \varepsilon) = \frac{2x^{1-\varepsilon}\varepsilon^2 \sin\varepsilon}{\pi}$$

$$\times \int_{-1/\varepsilon}^{\infty} (1+v\varepsilon) \left[(x^2(1+(1+t\varepsilon)^2(1+v\varepsilon)^2) + \varepsilon^2v^2 + 4(1+v\varepsilon)\sin^2(\varepsilon/2)\right]$$

$$\times \varepsilon^{-3} \left[x^2(v+t)^2 + v^2 + 4(1+v\varepsilon)\frac{1}{\varepsilon^2}\sin^2(\varepsilon/2) + 2x^2tv(v+t)\varepsilon + x^2tv\varepsilon^2\right]^{-3/2}$$

$$\times \left[x^2(1+(1+\varepsilon t)(1+\varepsilon v))^2 + \varepsilon^2v^2 + 4(1+v\varepsilon)\sin^2(\varepsilon/2)\right]^{-3/2} dv. \tag{3.48}$$

In view of (3.32) $\varepsilon I(x, x(1+t\varepsilon), \varepsilon)$ converges uniformly in ε, and we have

$$\lim_{\varepsilon \to 0+} \varepsilon I(x, x(1+t\varepsilon), \varepsilon) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dv}{((x^2+1)v^2 + 2x^2tv + x^2t^2 + 1)^{3/2}}$$

$$= \frac{1}{\pi} \frac{\sqrt{x^2+1}}{x^2t^2+x^2+1}, \tag{3.49}$$

where formula 2.2.9.22 from Prudnikov et al. [1] is applied, and we obtain (3.47). Observing that for all $x > 0$ the equality

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x\sqrt{x^2+1}dt}{x^2t^2+x^2+1} = 1$$

being directly proved and applying the generalized Minkowski inequality (1.10) we obtain the desired estimate of the difference $||(I_\varepsilon g) - f||_{L_p}$. In fact, using representation (3.39) we have

$$||(I_\varepsilon g) - f||_{L_p} = \left|\left| \int_{-\infty}^{\infty} H(t + 1/\varepsilon) x\varepsilon I(x, x(1+t\varepsilon), \varepsilon) f(x(1+t\varepsilon)) dt \right.\right.$$
$$\left.\left. - \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x\sqrt{x^2+1} f(x)}{x^2t^2+x^2+1} dt \right|\right|_{L_p}$$
$$\leq \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{t^2+1} \left|\left| H\left(\frac{\sqrt{x^2+1}}{x}t + \frac{1}{\varepsilon}\right) \pi\sqrt{x^2+1}(t^2+1)\varepsilon \right.\right.$$
$$\left.\left. \times I\left(x, x\left(1 + \frac{\sqrt{x^2+1}}{x}t\varepsilon\right), \varepsilon\right) f\left(x\left(1 + \frac{\sqrt{x^2+1}}{x}t\varepsilon\right)\right) - f(x) \right|\right|_{L_p}, \tag{3.50}$$

where $H(x)$ is the Heaviside function. Thus the right-hand side of (3.50) tends to zero when $\varepsilon \to 0+$ due to the Lebesgue theorem and limit's relation (3.47). So we established (3.37) and inversion formula (3.36) for L_p-functions. The existence of the limit almost everywhere on $\mathbf{R}_+$ follows from the radial property of the Poisson kernel (1.14). Theorem 3.2 is proved. •

From estimate (3.46) the inequality

$$||\,(I_\varepsilon g)\,||_{L_p} < C||\,(Ig)\,||_{L_p} \tag{3.51}$$

holds in view of (3.36). Theorem 3.2 shows that $MF[f] \equiv 0$, for $f(y) \in L_p(\mathbf{R}_+)$, $1 < p < \infty$, iff $f(y) \equiv 0$. So, in the space $MF(L_p)$ one can introduce a norm by the equality

$$||g||_{MF(L_p)} = ||f||_{L_p}, \; g = MF[f]. \tag{3.52}$$

It is easily find, that the space $MF(L_p)$ is a Banach space with the norm (3.52) and is isometric to L_p.

The main result of this section is to describe the space $MF(L_p)$ in terms of the operator I_ε defined in (3.19).

Theorem 3.3. *In order to $g(\tau) \in MF(L_p)$, $1 < p < \infty$, it is necessary and sufficiently that the following conditions hold:*

$$\text{l.i.m.}_{\varepsilon\to 0+} \,(I_\varepsilon g)\,(x) \in L_p(\mathbf{R}_+), \tag{3.53}$$

and $g(\tau) \in L_r(\mathbf{R}_+; e^{-\alpha\tau})$, $\alpha > 0$, $1 \le r \le \infty$ *at the necessity part and* $g(\tau) \in L_r(\mathbf{R}_+)$, $1 \le r \le \infty$ *at the sufficiency part.*

Proof. The necessity of condition (3.53) follows from the previous Theorem 3.2 and from estimate (3.18). Let us prove the sufficiency. Let $g(\tau) \in L_r(\mathbf{R}_+)$ and condition (3.53) holds. We show that in this case there is a function $f(x) \in L_p(\mathbf{R}_+)$, such that the equality

$$g(\tau) = MF[f](\tau) \tag{3.54}$$

takes place. From (3.53) we conclude that the function $(I_\varepsilon g)(x)$ belongs $L_p(\mathbf{R}_+)$ for sufficiently small $\varepsilon \in (0,1)$ and one can evaluate the composition

$$MF[(I_\varepsilon g)](\tau) = \frac{\pi}{2}\int_0^\infty P_{-1/2+i\tau/2}(2y^2+1)\,(I_\varepsilon g)(y)dy. \tag{3.55}$$

At least for smooth functions with compact support on $\mathbf{R}_+$, the set of whose is dense in $L_r(\mathbf{R}_+)$ substituting (3.19) in equality (3.55) and changing the order of integration by the Fubini theorem, we have the relation

$$MF[(I_\varepsilon g)](\tau) \equiv g_\varepsilon(\tau) = \int_0^\infty M(\beta,\tau,\varepsilon)g(\beta)d\beta, \tag{3.56}$$

where

$$M(\beta,\tau,\varepsilon) = \frac{\beta\sinh((\pi-\varepsilon)\beta)}{2\cosh^2(\pi\beta/2)}\int_0^\infty y^{1-\varepsilon}P_{-1/2+i\tau/2}(2y^2+1)P_{-1/2+i\beta/2}(2y^2+1)dy. \tag{3.57}$$

Let us treat integral (3.57). First observe that the uniform inequality

$$|M(\beta,\tau,\varepsilon)| \le C_\varepsilon \beta e^{(\pi/2-\delta_1)\tau+(\pi/2-\varepsilon-\delta_2)\beta}$$
$$\times\int_0^\infty y^{1-\varepsilon}P^2_{-1/2}(2y^2+1)dy, \tag{3.58}$$

is true from estimate (3.16), where $0 < \varepsilon_0 < \varepsilon < 1$, $C_\varepsilon > 0$ is a constant and there exist such numbers $\delta_1,\ \delta_2 \in (0,\pi/2)$ that $\pi/2 - \alpha/r < \delta_1 < \pi/2$, $\pi/2 - \varepsilon < \delta_2 < \pi/2$. Hence we see that from the generalized Minkowski inequality (1.10) the integral operator of g in the right-hand side of (3.56) is bounded on the space $L_r(\mathbf{R}_+; e^{-\alpha\tau})$. Now let us find the representation of the kernel $M(\beta,\tau,\varepsilon)$. For this substitute integral (3.2) into (3.57) and deduce the equality

$$M(\beta,\tau,\varepsilon) = \frac{2\beta\sinh((\pi-\varepsilon)\beta)}{\pi^2}\frac{\cosh(\pi\tau/2)}{\cosh(\pi\beta/2)}$$
$$\times\int_0^\infty y^{1-\varepsilon}\int_0^\infty J_0(yu)K_{i\tau}(u)du\int_0^\infty J_0(yv)K_{i\beta}(v)dvdy. \tag{3.59}$$

Changing the order of integration in (3.59), we observe that the inside integrals with respect to u and v are absolutely convergent and uniformly in $y \in [0,\lambda]$ due to estimate (1.100) and inequality (3.33). Thus the integral (3.59) can be written in the form

$$M(\beta,\tau,\varepsilon) = \frac{2\beta\sinh((\pi-\varepsilon)\beta)}{\pi^2}\frac{\cosh(\pi\tau/2)}{\cosh(\pi\beta/2)}$$

$$\times \lim_{\lambda \to +\infty} \int_0^\infty \int_0^\infty K_{i\tau}(u) K_{i\beta}(v) du dv \int_0^\lambda y^{1-\varepsilon} J_0(yu) J_0(yv) dy. \tag{3.60}$$

The polar coordinates give

$$M(\beta, \tau, \varepsilon) = \frac{2\beta \sinh((\pi - \varepsilon)\beta)}{\pi^2} \frac{\cosh(\pi\tau/2)}{\cosh(\pi\beta/2)}$$

$$\times \lim_{\lambda \to +\infty} \int_0^{\pi/2} d\varphi \int_0^\infty K_{i\tau}(r\cos\varphi) K_{i\beta}(r \sin\varphi) r^{\varepsilon - 1} dr$$

$$\times \int_0^{r\lambda} y^{1-\varepsilon} J_0(y \cos\varphi) J_0(y \sin\varphi) dy. \tag{3.61}$$

Let us treat the last integral by y. We have

$$\int_0^{r\lambda} y^{1-\varepsilon} J_0(y\cos\varphi) J_0(y \sin\varphi) dy$$

$$= \int_0^1 y^{1-\varepsilon} J_0(y\cos\varphi) J_0(y \sin\varphi) dy$$

$$+ \int_1^{r\lambda} y^{1-\varepsilon} J_0(y\cos\varphi) J_0(y \sin\varphi) dy. \tag{3.62}$$

By the second mean value theorem the second integral in the right-hand side of (3.62) is equal to

$$\int_1^{r\lambda_1} y J_0(y\cos\varphi) J_0(y\sin\varphi) dy, \ \ \lambda_1 < \lambda. \tag{3.63}$$

Making use formula 1.8.3.10 from Prudnikov et al. [2] we obtain

$$\int_1^{r\lambda_1} y J_0(y\cos\varphi) J_0(y \sin\varphi) dy$$

$$= \frac{r\lambda_1 \left[\cos\varphi J_1(r\lambda_1 \cos\varphi) J_0(r\lambda_1 \sin\varphi) - \sin\varphi J_1(r\lambda_1 \sin\varphi) J_0(r\lambda_1\cos\varphi)\right]}{\cos^2\varphi - \sin^2\varphi}$$

$$- \frac{\cos\varphi J_1(\cos\varphi) J_0(\sin\varphi) - \sin\varphi J_1(\sin\varphi) J_0(\cos\varphi)}{\cos^2\varphi - \sin^2\varphi}. \tag{3.64}$$

To ensure the validity of the passing to the limit by $\lambda_1 \to \infty$ under the integral sign of (3.61) it is sufficient to consider the contribution of the first term of (3.64). That is, we estimate the integral

$$\int_0^{\pi/2} d\varphi \int_0^\infty K_{i\tau}(r\cos\varphi) K_{i\beta}(r\sin\varphi) r^{\varepsilon-1}$$

$$\times \frac{r\lambda_1 \left[\cos\varphi J_1(r\lambda_1\cos\varphi) J_0(r\lambda_1 \sin\varphi) - \sin\varphi J_1(r\lambda_1\sin\varphi) J_0(r\lambda_1 \cos\varphi)\right]}{\cos^2\varphi - \sin^2\varphi} dr. \tag{3.65}$$

Dividing the outside integral by φ into three parts by taking some fixed number $\xi \in (0, \pi/4)$, we obtain

$$\left[\int_0^{\pi/4-\xi} + \int_{\pi/4-\xi}^{\pi/4+\xi} + \int_{\pi/4+\xi}^{\pi/2}\right] d\varphi \int_0^\infty K_{i\tau}(r\cos\varphi) K_{i\beta}(r\sin\varphi) r^{\varepsilon-1}$$

$$\times \frac{r\lambda_1 [\cos\varphi J_1(r\lambda_1 \cos\varphi) J_0(r\lambda_1 \sin\varphi) - \sin\varphi J_1(r\lambda_1 \sin\varphi) J_0(r\lambda_1 \cos\varphi)]}{\cos^2(\varphi) - \sin^2(\varphi)} dr$$

$$= I_1 + I_2 + I_3. \tag{3.66}$$

Let us estimate the integral I_1. For this use formula 2.16.33.1 from Prudnikov et al. [2] to obtain

$$\int_0^\infty K_{i\tau}(r\cos\varphi) K_{i\beta}(r\sin\varphi) r^{\varepsilon-1} dr$$

$$= 2^{\varepsilon-3} \frac{\sin^{-\varepsilon}\varphi \cot^{-i\tau}\varphi}{\Gamma(\varepsilon)} \left| \Gamma\left(\frac{\varepsilon + i(\tau+\beta)}{2}\right) \Gamma\left(\frac{\varepsilon + i(\tau-\beta)}{2}\right) \right|^2$$

$$\times {}_2F_1\left(\frac{\varepsilon + i(\tau+\beta)}{2}, \frac{\varepsilon - i(\tau-\beta)}{2}; \varepsilon; 1 - \cot^2\varphi\right), \ \varepsilon > 0. \tag{3.67}$$

Accounting the simple inequality (1.147) for the Macdonald function, asymptotic of the Gauss hypergeometric function at infinity (1.86) and inequality for the Bessel function (3.33) we find that

$$|I_1| < C \int_0^{\pi/4-\xi} \frac{d\varphi}{\sqrt{\sin\varphi}} \int_0^\infty K_0(r\cos\varphi) K_0(r\sin\varphi) r^{\varepsilon-1} dr$$

$$< C_\varepsilon \int_0^{\pi/4-\xi} {}_2F_1\left(\frac{\varepsilon}{2}, \frac{\varepsilon}{2}; \varepsilon; 1 - \cot^2\varphi\right) \frac{d\varphi}{\sin^{\varepsilon+1/2}\varphi}$$

$$< A \int_0^{\pi/4-\xi} \frac{d\varphi}{\sqrt{\sin\varphi}} < \infty, \tag{3.68}$$

where C, C_ε, A are absolute positive constants which do not depend from λ_1. Similarly we estimate the integral I_3. Concerning the integral I_2 its estimation can be accounted by the behavior of the integrand at the neighborhood of the point $\varphi = \pi/4$. Indeed, we obtain that

$$|I_2| < C_\xi \int_{\pi/4-\xi}^{\pi/4+\xi} \frac{d\varphi}{\sin\varphi + \cos\varphi} < \infty. \tag{3.69}$$

Thus we established the possibility to pass to the limit under the sign of the iterated integral (3.61). Using formula 2.12.31.1 from Prudnikov et al. [2]

$$\int_0^\infty y^{1-\varepsilon} J_0(yr\cos\varphi) J_0(yr\sin\varphi) dy = \frac{2^{1-\varepsilon} r^{\varepsilon-2}}{(\cos\varphi + \sin\varphi)^{2-\varepsilon}} \frac{\Gamma(1-\varepsilon/2)}{\Gamma(\varepsilon/2)}$$

$$\times {}_2F_1\left(1 - \frac{\varepsilon}{2}, \frac{1}{2}; 1; \frac{2\sin 2\varphi}{(\sin\varphi + \cos\varphi)^2}\right) \tag{3.70}$$

and integral (3.67) we have the representation

$$M(\beta, \tau, \varepsilon) = \frac{\beta \sinh((\pi - \varepsilon)\beta) \Gamma(1 - \varepsilon/2)}{2\pi^2 \Gamma(\varepsilon/2) \Gamma(\varepsilon)}$$

$$\times\left|\Gamma\left(\frac{\varepsilon+i(\tau+\beta)}{2}\right)\Gamma\left(\frac{\varepsilon+i(\tau-\beta)}{2}\right)\right|^2$$

$$\times\frac{\cosh(\pi\tau/2)}{\cosh(\pi\beta/2)}\int_0^{\pi/2}\sin^{-\varepsilon}\varphi\cot^{-i\tau}\varphi\,{}_2F_1\left(\frac{\varepsilon+i(\tau+\beta)}{2},\frac{\varepsilon-i(\tau-\beta)}{2};\varepsilon;1-\cot^2\varphi\right)$$

$$\times{}_2F_1\left(1-\frac{\varepsilon}{2},\frac{1}{2};1;\frac{2\sin 2\varphi}{(\sin\varphi+\cos\varphi)^2}\right)\frac{d\varphi}{(\cos\varphi+\sin\varphi)^{2-\varepsilon}}. \tag{3.71}$$

Let us substitute this value into composition (3.56) preliminary by changing the variable $\tan\varphi = u$ in integral (3.71). Hence it implies that composition (3.56) is equal to

$$g_\varepsilon(\tau)=\frac{\Gamma(1-\varepsilon/2)}{2\pi^2\Gamma(\varepsilon)\Gamma(\varepsilon/2)}\int_0^\infty\left|\Gamma\left(\frac{\varepsilon+i(\tau+\beta)}{2}\right)\Gamma\left(\frac{\varepsilon+i(\tau-\beta)}{2}\right)\right|^2$$

$$\times\frac{\cosh(\pi\tau/2)}{\cosh(\pi\beta/2)}\beta\sinh((\pi-\varepsilon)\beta)M_1(\beta,\tau,\varepsilon)g(\beta)d\beta, \tag{3.72}$$

where

$$M_1(\beta,\tau,\varepsilon)=\int_0^\infty{}_2F_1\left(\frac{\varepsilon+i(\tau+\beta)}{2},\frac{\varepsilon-i(\tau-\beta)}{2};\varepsilon;1-\frac{1}{u^2}\right)$$

$$\times{}_2F_1\left(1-\frac{\varepsilon}{2},\frac{1}{2};1;\frac{4u}{(u+1)^2}\right)\frac{u^{i\tau-\varepsilon}}{(u+1)^{2-\varepsilon}}du. \tag{3.73}$$

After the substitution $\beta=\tau+\varepsilon t$ we obtain

$$g_\varepsilon(\tau)=\frac{\Gamma(1-\varepsilon/2)}{2\pi^2\Gamma(\varepsilon)\Gamma(\varepsilon/2)}\int_0^\infty H(\tau+\varepsilon t)\left|\Gamma\left(i\tau+\frac{\varepsilon}{2}(1+it)\right)\Gamma\left(\frac{\varepsilon(1-it)}{2}\right)\right|^2$$

$$\times\frac{\cosh(\pi\tau/2)(\tau+\varepsilon t)}{\cosh(\pi(\tau+\varepsilon t)/2)}\sinh[(\pi-\varepsilon)(\tau+\varepsilon t)]M_1(\tau+\varepsilon t,\tau,\varepsilon)g(\tau+\varepsilon t)dt, \tag{3.74}$$

where $H(x)$ is the Heaviside function. As we know from estimate (3.58) the kernel of (3.74) is a bounded function of three variables. Further, let us prove that the following limit relation is true

$$\lim_{\varepsilon\to 0+}\varepsilon M(\tau+\varepsilon t,\tau,\varepsilon)=\frac{1}{\pi}\frac{1}{t^2+1},\ \tau>0,\ t\in\mathbf{R}. \tag{3.75}$$

Indeed, using the self-transformation formula (1.54) for Gauss's hypergeometric function we have the representation for integral (3.73) as

$$M_1(\tau+\varepsilon t,\tau,\varepsilon)=\int_0^\infty{}_2F_1\left(-i\tau+\frac{\varepsilon(1-it)}{2},\frac{\varepsilon(1+it)}{2};\varepsilon;1-\frac{1}{u^2}\right)$$

$$\times{}_2F_1\left(\frac{\varepsilon}{2},\frac{1}{2};1;\frac{4u}{(u+1)^2}\right)\frac{|u-1|^{\varepsilon-1}u^{(3\tau+2\varepsilon t)i-\varepsilon}}{u+1}du. \tag{3.76}$$

For a fixed number $0 < \mu < 1$ divide integral (3.76) on the three parts

$$M_1(\tau+\varepsilon t,\tau,\varepsilon) = \left[\int_0^{1-\mu} + \int_{1-\mu}^{1+\mu} + \int_{1+\mu}^{\infty}\right]$$

$$\times {}_2F_1\left(-i\tau + \frac{\varepsilon(1-it)}{2}, \frac{\varepsilon(1+it)}{2}; \varepsilon; 1-\frac{1}{u^2}\right)$$

$$\times {}_2F_1\left(\frac{\varepsilon}{2}, \frac{1}{2}; 1; \frac{4u}{(u+1)^2}\right) \frac{|u-1|^{\varepsilon-1}u^{(3\tau+2\varepsilon t)i-\varepsilon}}{u+1}du$$

$$\equiv I_1 + I_2 + I_3. \tag{3.77}$$

It is easily seen that integrals εI_1 and εI_3 tend to zero as $\varepsilon \to 0+$, because these integrals are absolutely and uniformly convergent by $\varepsilon \in [0,1]$. Concerning the middle integral use the mean value theorem and arrive to the relation

$$\varepsilon I_2 = {}_2F_1\left(-i\tau + \frac{\varepsilon(1-it)}{2}, \frac{\varepsilon(1+it)}{2}; \varepsilon; 1-\frac{1}{\mu_1^2}\right)$$

$$\times {}_2F_1\left(\varepsilon/2, 1/2; 1; \frac{4\mu_1}{(\mu_1+1)^2}\right) \frac{\mu_1^{(3\tau+2\varepsilon t)i-\varepsilon}}{\mu_1+1}\varepsilon \int_{1-\mu}^{1+\mu} |\, u-1\,|^{\varepsilon-1}\, du, \tag{3.78}$$

where $\mu_1 \in (1-\mu, 1+\mu)$. Let us put now $\mu = \varepsilon$ at (3.78). After calculation of the integral and using property (1.51) of the Gauss hypergeometric function, we find $\lim_{\varepsilon\to 0+} \varepsilon I_2 = 1$. Hence relation (3.75) can be deduced by using the reduction formula (1.23) for the gamma-function .

As in Theorem 3.2, we derive the following estimates for norms of function (3.56) $g_\varepsilon(\tau)$ in the space $L_r(\mathbf{R}_+; e^{-\alpha\tau}) \subset L_r(\mathbf{R}_+)$

$$||g_\varepsilon(\tau) - g(\tau)||_{L_r(\mathbf{R}_+;e^{-\alpha\tau})}$$

$$\leq \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{t^2+1}||g(\tau+\varepsilon t)\varepsilon\pi(t^2+1)M(\tau+\varepsilon t,\tau,\varepsilon)$$

$$-g(\tau)||_{L_r(\mathbf{R}_+)}dt \to 0,\ \varepsilon \to 0+. \tag{3.79}$$

But, on the other hand, since the operator $MF[f]$ is bounded from $L_p(\mathbf{R}_+)$ into $L_p(\mathbf{R}_+; e^{-\alpha\tau})$-space, where $1 \leq p < \infty$, there exists the limit in $L_p(\mathbf{R}_+; e^{-\alpha\tau})$-norm

$$\text{l.i.m.}_{\varepsilon\to 0+} MF[I_\varepsilon g](\tau) = MF\left[\text{l.i.m.}_{\varepsilon\to 0+}(I_\varepsilon g)\right](\tau) = MF[f](\tau), \tag{3.80}$$

where $f = Ig \in L_p(\mathbf{R}_+)$. Since the operator $MF[I_\varepsilon g]$ converges in the norm $L_r(\mathbf{R}_+; e^{-\alpha\tau})$ too, then the limit functions must coincide almost everywhere on $\mathbf{R}_+$. Thus, from equality (3.80) we arrive to (3.54). Theorem 3.3 is completely proved. •

3.2 The composition theorem of inversion

We already mentioned in previous Chapter 2 and at the beginning of this chapter the importance of the Kontorovich-Lebedev transform (2.1) in connection with so-called problem of *the composition structure* of the index transforms. Indeed, using known mapping properties of the Mellin convolution type transforms and the K-L transform we establish new theorems for familiar index transforms and moreover, we construct some new index transforms with hypergeometric type of special functions as the kernels. Such approach has been developed recently in Yakubovich and Luchko [2] although it should be noted that many important results were obtained earlier (see for example, Yakubovich [1], Vu Kim Tuan et al. [1], Vu Kim Tuan [3]-[5], Yakubovich [3]-[4], [7]-[8], Yakubovich et al. [1], Samko et al. [1], Pestun [1]-[2], Ryko [1]-[2]).

Here as the corollary from theorems of the first section we study composition properties of the introduced Mehler-Fock transform (3.1) based as you already conjectured on the K-L transform (2.1) and the Hankel transform (1.225) of zero index. Taking into account integral representation (3.2) let us consider a slightly different Mehler-Fock transform, precisely

$$\frac{1}{\cosh(\pi\tau/2)}MF[\sqrt{y}f(y)](\tau) \equiv g(\tau) = \frac{\pi}{2\cosh(\pi\tau/2)}$$

$$\times \int_0^\infty P_{-1/2+i\tau/2}(2y^2+1)\sqrt{y}f(y)dy, \quad \tau \geq 0. \tag{3.81}$$

To write the mentioned composition of integral transforms the main problem is to interchange the order of integration by Fubini's theorem in the following iterated integral (after substitution (3.2) into (3.81))

$$g(\tau) = \int_0^\infty \sqrt{y}f(y)\int_0^\infty J_0(yu)K_{i\tau}(u)dudy. \tag{3.82}$$

First, using estimate (3.16) show that integral (3.81) is absolutely convergent under conditions $f(x) \in L_{\nu,p}(\mathbf{R}_+)$, $\nu \in \mathbf{R}$, $p \geq 1$, where the range of parameter ν we define below. Indeed, we have

$$|g(\tau)| \leq Ce^{-\delta\tau}\int_0^\infty P_{-1/2}(2y^2+1)\sqrt{y}|f(y)|dy$$

$$\leq C_1e^{-\delta\tau}\left[\int_0^1 \sqrt{y}|f(y)|dy + \int_1^\infty \frac{1}{\sqrt{y}}|f(y)|dy\right]$$

$$\leq C_1e^{-\delta\tau}\left[\left(\int_0^1 y^{(3/2-\nu)q-1}dy\right)^{1/q} + \left(\int_1^\infty y^{(1/2-\nu)q-1}dy\right)^{1/q}\right]||f||_{\nu,p}, \tag{3.83}$$

where $q = p/(p-1)$. Furthermore, as is evident for the convergence of integrals at the right-hand side of (3.83) that are deduced after applying the Hölder inequality like (1.21) we have to choose the parameter ν from $1/2 < \nu < 3/2$. Hence observe that the

Mehler-Fock transform (3.81) is a bounded operator from the space $L_{\nu,p}(\mathbf{R}_+)$, $1/2 < \nu < 3/2$ into any space $L_r(\mathbf{R}_+)$ and one can write the equality

$$g(\tau) = \frac{\pi}{2\cosh(\pi\tau/2)} \text{l.i.m.}_{\varepsilon\to+0} \int_0^{1/\varepsilon} P_{-1/2+i\tau/2}(2y^2+1)\sqrt{y}f(y)dy \tag{3.84}$$

meaning the above limit by the L_r-norm. Now from (3.82) we immediately establish the equivalent equality as

$$g(\tau) = \text{l.i.m.}_{\varepsilon\to+0} \int_0^{1/\varepsilon} \sqrt{y}f(y) \int_0^\infty J_0(yu)K_{i\tau}(u)dudy. \tag{3.85}$$

Further, for each $\varepsilon > 0$ one can change the order of integration in the iterated integral (3.85) provided by conditions $L_{\nu,p}(\mathbf{R}_+)$, $1 < \nu < 3/2$ as it is not difficult to conclude from inequalities (1.100) and (3.33). Thus we obtain

$$g(\tau) = \text{l.i.m}_{\varepsilon\to+0} \int_0^\infty \frac{K_{i\tau}(u)}{\sqrt{u}} \int_0^{1/\varepsilon} \sqrt{yu}J_0(yu)f(y)dydu. \tag{3.86}$$

So we already deduced the composition representation of the Mehler-Fock transform (3.81) through the K-L transform (2.1) and the Hankel transform (1.225) with the power multiplier $1/\sqrt{u}$. According to Theorem 1.21 the Hankel transform (1.225) of zero index is a bounded operator from $L_{\nu,p}(\mathbf{R}_+)$, $1 < \nu < 3/2$ into $L_{1-\nu,p}(\mathbf{R}_+)$. Thus $u^{-1/2}[J_0 f](u) \in L_{1/2-\nu,p}(\mathbf{R}_+)$ and due to Lemma 2.3 the K-L operator (2.1) exists from the mentioned Hankel transform under condition $1 < \nu < 3/2$. Hence meaning the inner limit by the norm of the space $L_{1/2-\nu,p}(\mathbf{R}_+)$ one can pass to the limit under the Kontorovich-Lebedev operator. Consequently, we obtain the following composition theorem.

Theorem 3.4. *The Mehler-Fock transform* (3.81) *is a bounded operator from the space* $L_{\nu,p}(\mathbf{R}_+)$, $1 < \nu < 3/2$, $p \geq 1$ *into the space* $L_r(\mathbf{R}_+)$, $r \geq 1$ *and its composition representation through the K-L transform* (2.1) *and the Hankel transform* (1.225) *of zero index for all* $\tau \geq 0$ *is true*

$$g(\tau) = K_{i\tau}\left[\frac{1}{\sqrt{x}}[J_0 f](x)\right]. \tag{3.87}$$

In view of this conclude that the Mehler-Fock transform (3.81) belongs to the space $KL(L_{1/2-\nu,p})$ (see (2.13)). Moreover, using Theorem 2.2 apply inversion formula for the K-L transform like (2.14) to establish the equality

$$[J_0 f](x) = \text{l.i.m.}_{\varepsilon\to+0} \frac{2}{\pi^2 x^{1/2-\varepsilon}} \int_0^\infty \tau\sinh((\pi-\varepsilon)\tau)K_{i\tau}(x)g(\tau)d\tau, \tag{3.88}$$

where the limit is meant by the norm of the space $L_{1-\nu,p}(\mathbf{R}_+)$. In order to invert completely the Mehler-Fock transform (3.81) apply again Theorem 1.21, namely the

Mellin transform relation (1.226) to obtain the inversion formula for the Hankel transform (1.225) by means of the Mellin-Parseval equality (1.214) (see details in Titchmarsh [1], Yakubovich and Luchko [2]). Thus accounting that the Hankel transform has a symmetric inversion we arrive

$$f(x) = \left[J_0 \text{l.i.m.}_{\varepsilon\to+0} \frac{2}{\pi^2 x^{1/2-\varepsilon}} \int_0^\infty \tau \sinh((\pi-\varepsilon)\tau) K_{i\tau}(x) g(\tau) d\tau\right]$$

$$= \text{l.i.m.}_{\varepsilon\to+0} \left[J_0 \frac{2}{\pi^2 x^{1/2-\varepsilon}} \int_0^\infty \tau \sinh((\pi-\varepsilon)\tau) K_{i\tau}(x) g(\tau) d\tau\right]. \tag{3.89}$$

Here we carry out the limit in (3.89) according to Theorem 1.21 since the inverse Hankel transform is one-to-one bounded operator from $L_{1-\nu,p}(\mathbf{R}_+)$ into $L_{\nu,p}(\mathbf{R}_+)$ and the last limit is meant by the norm of $L_{\nu,p}$, $1 < p \le 2$. Writing the iterated integral that follows from (3.89) we continue

$$f(x) = \text{l.i.m.}_{\varepsilon\to+0} \frac{2\sqrt{x}}{\pi^2} \int_0^\infty y^\varepsilon J_0(xy)$$

$$\times \int_0^\infty \tau \sinh((\pi-\varepsilon)\tau) K_{i\tau}(y) g(\tau) d\tau dy. \tag{3.90}$$

Then change the order of integration by the Fubini theorem as it is possible under estimates (1.100), (3.33) and (3.83). Thus we finally establish the inversion formula like (3.19) for the Mehler-Fock transform (3.81) but by the index of Gauss's hypergeometric function (1.47) after calculation of the inner integral by formula (1.101) . Precisely, we have

$$f(x) = \text{l.i.m.}_{\varepsilon\to+0} \frac{2^{\varepsilon-1}\sqrt{x}}{\pi} \int_0^\infty \Gamma\left(\frac{\varepsilon+1+i\tau}{2}\right) \Gamma\left(\frac{\varepsilon+1-i\tau}{2}\right)$$

$$\times \tau \sinh((\pi-\varepsilon)\tau) {}_2F_1\left(\frac{\varepsilon+1+i\tau}{2}, \frac{\varepsilon+1-i\tau}{2}; 1; -x^2\right) g(\tau) d\tau. \tag{3.91}$$

Theorem 3.5. *Under conditions of Theorem* 3.4 *for* $1 < p \le 2$ *the Mehler-Fock transform* (3.81) *has inversion formula* (3.91).

Remark 3.1. Letting $\varepsilon = 0$ at (3.91) , and using formulae (1.55)-(1.56) after substitution integral (3.81) we easily come to expansion (1.233) up to change of variables and functions.

3.3 The generalized Mehler-Fock transform

In this section we introduce *the generalized Mehler-Fock transform* with the associated Legendre function of the first kind $P_\nu^\mu(z)$ represented by formulae (1.55)-(1.56) with complex $\mu \neq 0$ in general case. In fact, this kernel we derive for example, from integral (1.101), putting there $\alpha + \mu = 1$. Namely, we have the formula

$$\int_0^\infty y^{-\mu} J_\mu(xy) K_{i\tau}(y) dy = 2^{-\mu-1} x^\mu \frac{\pi}{\Gamma(\mu+1)\cosh(\pi\tau/2)}$$

$$\times {}_2F_1\left(\frac{1+i\tau}{2}, \frac{1-i\tau}{2}; \mu+1; -x^2\right)$$

$$= \frac{2^{-\mu-1}x^{2\mu}}{(x^2+1)^{\mu/2}} \frac{\pi}{\cosh(\pi\tau/2)} P^{-\mu}_{-1/2+i\tau/2}(2x^2+1). \tag{3.92}$$

Similarly define the generalized Mehler-Fock transform by the integral

$$g(\tau) = \frac{2^{-\mu-1}\pi}{\cosh(\pi\tau/2)} \int_0^\infty \frac{y^{2\mu}}{(y^2+1)^{\mu/2}} P^{-\mu}_{-1/2+i\tau/2}(2y^2+1)\sqrt{y} f(y) dy, \ \tau \geq 0. \tag{3.93}$$

When $\mu = 0$ we immediately obtain the Mehler-Fock transform (3.81). Further, invoking with inequalities (1.100) and (3.33) for $\Re\mu \geq -1/2$ from representation (3.92) we have the uniform estimate of kind

$$\frac{2^{-\Re\mu-1}x^{2\Re\mu}}{(x^2+1)^{\Re\mu/2}} \frac{\pi}{\cosh(\pi\tau/2)} |P^{-\mu}_{-1/2+i\tau/2}(2x^2+1)|$$

$$\leq Ce^{-\delta\tau} x^{-1/2} \int_0^\infty y^{-\Re\mu-1/2} K_0(y\cos\delta) dy, \ 0 \leq \delta < \pi/2, \tag{3.94}$$

and from the asymptotic behavior of the Macdonald function (1.96)-(1.97) we observe the convergence of integral (3.94) for $-1/2 \leq \Re\mu < 1/2$. Thus we arrive to the inequality

$$|P^{-\mu}_{-1/2+i\tau/2}(2x^2+1)| \leq Ce^{\pi/2-\delta\tau} x^{-1/2-2\Re\mu}(x^2+1)^{\Re\mu/2}, \ \Re\mu < 1/2 \tag{3.95}$$

with the right-hand side by variable x as $O(x^{-1/2-2\Re\mu})$ as $x \to 0+$ and $O(x^{-1/2-\Re\mu})$ as $x \to \infty$. Although for our further considerations it is important slightly change estimate (3.95) when $x \in [0,1]$. Namely, turning to relation (3.92) use inequality $x^{-\mu}|J_\mu(x)| < C, \ -1/2 \leq \Re\mu, \ x > 0$ and our estimate becomes

$$\frac{2^{-\Re\mu-1}x^{2\Re\mu}}{(x^2+1)^{\Re\mu/2}} \frac{\pi}{\cosh(\pi\tau/2)} |P^{-\mu}_{-1/2+i\tau/2}(2x^2+1)|$$

$$\leq Ce^{-\delta\tau} x^{\Re\mu} \int_0^\infty K_0(y\cos\delta) dy \leq C_\delta x^{\Re\mu} e^{-\delta\tau}, \ 0 \leq \delta < \pi/2. \tag{3.96}$$

Therefore, considering $f(x) \in L_{\nu,p}(\mathbf{R}_+), p \geq 1$ and applying inequalities (3.95)-(3.96) we derive the estimate of the generalized Mehler-Fock transform (3.93) as

$$|g(\tau)| \leq Ce^{-\delta\tau} \left[\int_0^1 y^{\Re\mu}|f(y)|dy + \int_1^\infty |f(y)|dy\right], \ \tau \geq 0. \tag{3.97}$$

Making use the Hölder inequality find that

$$|g(\tau)| < Ce^{-\delta\tau}||f||_{\nu,p}\left[\left(\int_0^1 y^{(1+\Re\mu-\nu)q-1}dy\right)^{1/q} + \left(\int_1^\infty y^{(1-\nu)q-1}dy\right)^{1/q}\right] \tag{3.98}$$

and integrals at the right-hand side of (3.98) are finite ones provided that $1 < \nu < 1 + \Re\mu$. Thus we established the following theorem.

Theorem 3.6. *The generalized Mehler-Fock transform* (3.93) *is a bounded operator from the space* $L_{\nu,p}(\mathbf{R}_+)$, $1 < \nu < 1 + \Re\mu$, $0 < \Re\mu < 1/2$, $p \geq 1$ *into the space* $L_r(\mathbf{R}_+)$, $r \geq 1$ *and its composition representation through the K-L transform* (2.1) *and the Hankel transform* (1.225) *for all* $\tau \geq 0$ *is true*

$$g(\tau) = K_{i\tau}\left[x^{-\mu-1/2}[J_\mu f](x)\right]. \tag{3.99}$$

The proof of this theorem one can achieve by the above estimates and the Fubini theorem.

Similarly we obtain the inversion theorem for the generalized Mehler-Fock transform (3.93), sequentially inverting the respective Kontorovich-Lebedev and the Hankel transforms.

Theorem 3.7. *If* $f(x) \in L_{\nu,p}(\mathbf{R}_+)$, $1 < \nu < 1 + \Re\mu$, $0 < \Re\mu < 1/2$, $1 < p \leq 2$, *then for generalized Mehler-Fock transform* (3.93) *the inversion formula*

$$f(x) = \text{l.i.m.}_{\varepsilon\to+0}\frac{2^{\varepsilon+\mu-1}x^{1/2+\mu}}{\pi\Gamma(\mu+1)}\int_0^\infty \Gamma\left(\frac{\varepsilon+1+i\tau}{2+\mu}\right)\Gamma\left(\frac{\varepsilon+1-i\tau}{2+\mu}\right)$$

$$\times\tau\sinh((\pi-\varepsilon)\tau)\,{}_2F_1\left(\frac{\varepsilon+1+i\tau}{2+\mu}, \frac{\varepsilon+1-i\tau}{2+\mu}; \mu+1; -x^2\right)g(\tau)d\tau. \tag{3.100}$$

is valid.

Proof. In accordance with Theorems 1.21 and 2.3 combining with Theorem 3.6 we may conclude that the Mehler-Fock transform (3.93) belongs to the space $KL(L_{3/2+\Re\mu-\nu,p})$. Consequently, under conditions of this theorem one can invert the Hankel transform in (3.99) owing to Theorem 1.21, and we arrive to the iterated integral

$$f(x) = \text{l.i.m.}_{\varepsilon\to+0}\frac{2\sqrt{x}}{\pi^2}\int_0^\infty y^{\varepsilon+\mu}J_\mu(xy)\int_0^\infty \tau\sinh((\pi-\varepsilon)\tau)K_{i\tau}(y)g(\tau)d\tau dy. \tag{3.101}$$

Thus taking in mind the above estimates change the order of integration and use formula (1.101) that leads to (3.100). This completes the proof of Theorem 3.7. •

3.4 Parseval's equality

In this section in the same manner as for the Kontorovich-Lebedev transform (2.1) we deduce the Parseval relation for the Mehler-Fock transform (3.1). We refer the reader to similar questions in Lebedev [4], Lowndes [2], Yakubovich and Luchko [2].

Let us consider the Hilbert space of functions $L_{0,2}(\mathbf{R}_+)$ normed in accordance with (1.19) by

$$||f||_{L_{0,2}(\mathbf{R}_+)} = \left(\int_0^\infty |f(t)|^2 \frac{dt}{t}\right)^{1/2}. \tag{3.102}$$

Following further, introduce the corresponding inner product for two complex-valued functions $f(x)$, $g(x)$ as

$$\langle f, g\rangle = \int_0^\infty f(t)\overline{g(t)}\frac{dt}{t}. \tag{3.103}$$

Thus for each function $f(x) \in L_{0,2}(\mathbf{R}_+)$ the respective Mehler-Fock transform (3.1) is taken as the limit in mean of the integral

$$MF[f](\tau) = \frac{\pi}{2}\text{l.i.m.}_{N\to\infty}\int_{1/N}^{N} P_{-1/2+i\tau/2}(2y^2+1)f(y)dy. \tag{3.104}$$

This integral exists as an absolutely convergent one. Indeed, if $f(x) \in L_{0,2}(\mathbf{R}_+)$, then $f(x) \in L_{0,2}([1/N, N])$ for any number $N > 0$. Moreover the estimate

$$\int_{1/N}^{N} |f(y)|^2 \frac{dy}{y} < C\int_{1/N}^{N} |f(y)|^2 dy = C||f||^2_{L_2([1/N,N])} \tag{3.105}$$

is valid and consequently, $f(x) \in L_2([1/N, N])$. Therefore owing to inequality (3.16) and convergence of the integral

$$\int_{1/N}^{N} P_{-1/2}(2y^2+1)|f(y)|dy$$

$$\leq \left(\int_{1/N}^{N} P^2_{-1/2}(2y^2+1)dy\right)^{1/2} ||f||_{L_2([1/N,N])} < \infty, \tag{3.106}$$

we easily conclude that the Mehler-Fock transform (3.1) of the function $f_N = f(x)$, $x \in [1/N, N]$, $f(x) = 0$, $0 < x < 1/N$ exists for all $\tau \geq 0$. Furthermore, one can prove that the range of the Mehler-Fock transform (3.104) coincides with the weighted Hilbert space $L_2\left(\mathbf{R}_+; \frac{4}{\pi^2}\tau\tanh(\pi\tau/2)\right)$ with the norm

$$||h||_{L_2\left(\mathbf{R}_+;\frac{4}{\pi^2}\tau\tanh(\pi\tau/2)\right)} = \frac{2}{\pi}\left(\int_0^\infty \tau\tanh\left(\frac{\pi\tau}{2}\right)|h(\tau)|^2 d\tau\right)^{1/2} \tag{3.107}$$

and the limit in (3.104) is understood by means of the convergence by norm (3.107). Therefore, substituting the value of the Mehler-Fock transform (3.1) within the inner product $(MF[f](\tau), MF[g](\tau))$ being defined by the Hilbert space (3.107) we have formally

$$
\begin{aligned}
(MF[f], MF[g]) &= \frac{4}{\pi^2}\int_0^\infty \tau \tanh\left(\frac{\pi\tau}{2}\right) MF[f](\tau)\overline{MF[g](\tau)}d\tau \\
&= \frac{2}{\pi}\int_0^\infty \tau \tanh\left(\frac{\pi\tau}{2}\right) MF[f](\tau) \\
&\times \overline{\int_0^\infty P_{-1/2+i\tau/2}(2y^2+1)g(y)dy}d\tau \\
&= \int_0^\infty \overline{g(y)}dy \frac{2}{\pi}\int_0^\infty \tau \tanh\left(\frac{\pi\tau}{2}\right) \\
&\times P_{-1/2+i\tau/2}(2y^2+1) MF[f](\tau)d\tau \\
&= \int_0^\infty f(y)\overline{g(y)}\frac{dy}{y} = \langle f, g\rangle.
\end{aligned} \tag{3.108}
$$

The last equality in (3.108) is established due to (3.19), where we put formally $\varepsilon = 0$. Motivation of this can be given by the following theorem.

Theorem 3.8. *If* $g(x) \in L_1(\mathbf{R}_+; P_{-1/2}(2x^2+1))$ *and* $MF[f](\tau) \in L_1(\mathbf{R}_+; \tau\exp((\pi/2-\delta)\tau))$, $\delta \in [0, \pi/2)$, *then the Parseval equality is true*

$$
(MF[f], MF[g]) = \langle f, g\rangle. \tag{3.109}
$$

Proof. The proof of this theorem implies from the Fubini theorem that can be performed to apply under inequality (3.16) for the Legendre function and the above conditions. More precisely, observe the convergence of the iterated integral

$$
\int_0^\infty \tau \tanh\left(\frac{\pi\tau}{2}\right) |MF[f](\tau)| \int_0^\infty |P_{-1/2+i\tau/2}(2y^2+1)g(y)|dyd\tau
$$

$$
\leq C\int_0^\infty P_{-1/2}(2y^2+1)|g(y)|dy \int_0^\infty \tau\exp((\pi/2-\delta)\tau)|MF[f](\tau)|d\tau < \infty
$$

and confirm the desired result. Theorem 3.8 is proved. •

Letting $f = g$ at the equality (3.109) we obtain that $f \in L_{0,2}(\mathbf{R}_+)$ and $h(\tau) = MF[f](\tau) \in L_2\left(\mathbf{R}_+; \frac{4}{\pi^2}\tau\tanh(\pi\tau/2)\right)$. Furthermore,

$$
||f||_{L_{0,2}(\mathbf{R}_+)} = ||h||_{L_2\left(\mathbf{R}_+; \frac{4}{\pi^2}\tau\tanh(\pi\tau/2)\right)}. \tag{3.110}
$$

Similarly as in the case of the K-L transform take the space $C^{(2)}(\mathbf{R}_+)$ of the smooth functions of the order two with a compact support on $\mathbf{R}_+$. As result we arrive to the analog of Lemma 2.5 for the Mehler-Fock transform (3.1).

Lemma 3.2. *If the function $f(x) \in C^{(2)}(\mathbf{R}_+)$ and its support is a compact one, then the Mehler-Fock transform* (3.1) *belongs to the space $L_1(\mathbf{R}_+;\sqrt{\tau})$.*

Proof. It is not difficult to see that $MF[f](\tau)$ is continuous function of variable $\tau \in \mathbf{R}_+$. Further, use Theorem 1.8 of the asymptotic behavior of the Legendre function $P_{-1/2+i\tau}(2x^2+1)$ by index $\tau \to +\infty$. Indeed, invoking with formulae (1.155)-(1.157) as well as asymptotic behavior of the Bessel function (1.92) at infinity we obtain

$$MF[f](\tau) = \frac{1}{\sqrt{\tau}} O\left(\int_0^\infty e^{i\tau y} f(y)dy\right), \tau \to +\infty. \tag{3.111}$$

Hence we have the estimate like (2.51), namely

$$|MF[f](\tau)| < \frac{C}{\tau^{5/2}}, \tau > 0, \tag{3.112}$$

where $C > 0$ is an absolute constant. This completes the proof of Lemma 3.2. •

Corollary 3.1. *For functions $f(x)$ from the space $C^{(2)}(\mathbf{R}_+)$ the Parseval equality* (3.109) *is true.*

Proof. In fact, in this case we appeal to Theorem 3.2 as well as Theorem 1.8 that perform to pass to the limit under the sign of integral (3.19). Thus the equality

$$\frac{f(x)}{x} = \frac{2}{\pi}\int_0^\infty \tau \tanh\left(\frac{\pi\tau}{2}\right) P_{-1/2+i\tau/2}(2x^2+1) MF[f](\tau)d\tau \tag{3.113}$$

is valid and moreover, this integral is absolutely convergent. Hence one can change the order of integration like in the proof of Theorem 3.8 and to obtain the Parseval equality (3.109). Corollary 3.1 is proved. •

Let now $f(x)$ be an arbitrary function from the space $L_{0,2}(\mathbf{R}_+)$. Choose some sequence of functions from the space $C^{(2)}(\mathbf{R}_+)$ with the compact support that is convergent to the given function f by norm of the space $L_{0,2}(\mathbf{R}_+)$. Denote as in Chapter 2 through f_n the common term of this sequence and through symbol I_n the least segment which contains the support of the function f_n. Since the operator of the Mehler-Fock transform is linear one then from Corollary 3.1 the equality follows

$$\int_0^\infty |f_n(x) - f_m(x)|^2 \frac{dx}{x} = \frac{4}{\pi^2}\int_0^\infty \tau \tanh\left(\frac{\pi\tau}{2}\right) |MF[f_n](\tau) - MF[f_m](\tau)|^2 d\tau. \tag{3.114}$$

Indeed, the left-hand side of equality (3.114) tends to zero by $m, n \to \infty$. Therefore, the sequence $\{MF[f_n](\tau)\}$ is the Cauchy one. The completeness of the Hilbert space $L_2\left(\mathbf{R}_+; \frac{4}{\pi^2}\tau\tanh(\pi\tau/2)\right)$ means the existence of the function $h(\tau) \equiv MF[f] \in L_2\left(\mathbf{R}_+; \frac{4}{\pi^2}\tau\tanh(\pi\tau/2)\right)$ such that $MF[f_n](\tau) \to h(\tau)$ by the norm of this space. Since

$$MF[f_n](\tau) = \frac{\pi}{2}\int_{I_n} P_{-1/2+i\tau/2}(2y^2+1) f_n(y)dy, \tag{3.115}$$

then, integrating through by segment $[0, \tau]$ we obtain

$$\int_0^\tau MF[f_n](t)dt = \frac{\pi}{2}\int_{I_n} f_n(y)dy \int_0^\tau P_{-1/2+it/2}(2y^2+1)dt$$
$$= \int_0^\infty P(\tau, y)f_n(y)dy, \tag{3.116}$$

where

$$P(\tau, y) = \frac{\pi}{2}\int_0^\tau P_{-1/2+it/2}(2y^2+1)dt. \tag{3.117}$$

Let us consider the left-hand side of equality (3.116). As $MF[f_n](t)$ belongs to $L_2\left(\mathbf{R}_+; \frac{4}{\pi^2}t\tanh(\pi t/2)\right)$, consequently $MF[f_n](t) \in L_2([0;\tau])$. Since $MF[f_n](t) \to MF[f](t)$ by the norm of the space $L_2\left(\mathbf{R}_+; \frac{4}{\pi^2}t\tanh(\pi t/2)\right)$ and

$$\int_0^\tau |MF[f_n](t) - MF[f](t)|^2 dt < C||MF[f_n] - MF[f]||^2_{L_2\left(\mathbf{R}_+; \frac{4}{\pi^2}t\tanh(\pi t/2)\right)}, \tag{3.118}$$

then $MF[f_n] \to MF[f]$ by the norm $L_2([0;\tau])$. Hence by the Cauchy-Schwarz-Bunyakovskii inequality we have

$$\left|\int_0^\tau (MF[f_n](t) - MF[f](t))dt\right| \le \int_0^\tau |MF[f_n](t) - MF[f](t)|dt$$
$$\le \sqrt{\tau}||MF[f_n] - MF[f]||_{L_2([0;\tau])}. \tag{3.119}$$

Therefore,

$$\lim_{n\to\infty}\int_0^\tau MF[f_n](t)dt = \int_0^\tau MF[f](t)dt. \tag{3.120}$$

Similarly we establish the limit at the right-hand side of (3.116). Indeed, the function $f_n(x) \in L_{0,2}(\mathbf{R}_+)$ and invoking with inequality (1.21) we obtain the estimate

$$\int_0^\infty |P(\tau, y)f_n(y)|dy \le \left(\int_0^\infty y|P(\tau, y)|^2 dy\right)^{1/2} ||f_n||_{L_{0,2}(\mathbf{R}_+)}. \tag{3.121}$$

Thus one can show that for each $\tau > 0$ the function $P(\tau, y) \in L_{1,2}(\mathbf{R}_+)$. In fact, from representation (3.2) in view of (3.117) we have

$$\sqrt{x}P(\tau, x) = \int_0^\tau \cosh(\pi t/2)dt \int_0^\infty \sqrt{x}J_0(xy)K_{it}(y)dy. \tag{3.122}$$

As is evident from the asymptotic properties of the Bessel functions the integral (3.122) is convergent absolutely and uniformly by t. Therefore, we can integrate through by t in (3.122) and invoking with the second mean value theorem to write it as

$$\sqrt{x}P(\tau, x) = C_\tau \int_0^\infty \sqrt{x}J_0(xy)\int_0^\tau K_{it}(y)dydt$$
$$= C_\tau \int_0^\infty \sqrt{x}J_0(xy)K(\tau, y)dy, \tag{3.123}$$

where C_τ is some constant which does not depend upon x and the function $K(\tau, y)$ is defined by formula (2.56). Hence formula (3.123) can be written in slightly different form, namely

$$\sqrt{x}P(\tau, x) = C_\tau \int_0^\infty \sqrt{xy} J_0(xy) \frac{K(\tau, y)}{\sqrt{y}} dy. \tag{3.124}$$

This implies that the right-hand side of (3.124) is the Hankel transform (1.225) of zero index and according to (2.60) the function $K(\tau, x)/\sqrt{x}$ belongs to the space $L_2(\mathbf{R}_+)$ for each $\tau > 0$. Thus as is known in Titchmarsh [1] integral (3.124) is convergent in mean value, i.e. the following limit equality is true

$$\begin{aligned} g(\tau, x) &= \text{l.i.m.}_{N\to\infty} \int_0^N \sqrt{xy} J_0(xy) \frac{K(\tau, y)}{\sqrt{y}} dy \\ &= \text{l.i.m.}_{N\to\infty} g_N(\tau, x), \end{aligned} \tag{3.125}$$

by the norm of the space $L_2(\mathbf{R}_+)$. On the other hand, due to the estimate we have

$$\begin{aligned} &\left| \frac{\sqrt{x}P(\tau, x)}{C_\tau} - \int_0^N \sqrt{xy} J_0(xy) \frac{K(\tau, y)}{\sqrt{y}} dy \right| \\ &= \left| \int_N^\infty \sqrt{xy} J_0(xy) \frac{K(\tau, y)}{\sqrt{y}} dy \right| \\ &\leq A_\tau \int_N^\infty \frac{K_0(y)}{\sqrt{y}} dy \to 0, N \to \infty, \end{aligned} \tag{3.126}$$

where the positive constant A_τ does not depend upon x according to inequality (3.33). So we conclude that integral (3.124) is uniformly convergent too, and therefore the limit function $g(\tau, x)$ in (3.125) coincides with $\sqrt{x}P(\tau, x)/C_\tau$ and moreover, $\sqrt{x}P(\tau, x) \in L_2(\mathbf{R}_+)$ or $P(\tau, x) \in L_{1,2}(\mathbf{R}_+)$. From the relation $f_n \to f$ by the norm of $L_{0,2}(\mathbf{R}_+)$ and by the Cauchy-Schwarz-Bunyakovskii inequality we have that

$$\lim_{n\to\infty} \int_0^\infty P(\tau, y) f_n(y) dy = \int_0^\infty P(\tau, y) f(y) dy \tag{3.127}$$

and passing to the limit at the equality (3.116) obtain that

$$\int_0^\tau MF[f](t) dt = \int_0^\infty P(\tau, y) f(y) dy. \tag{3.128}$$

Since $MF[f](t) \in L_2\left(\mathbf{R}_+; \frac{4}{\pi^2} t \tanh(\pi t/2)\right)$, then $MF[f](t) \in L_2((0, N])$ and therefore $MF[f](t) \in L_1((0, N])$. Consequently, one can differentiate through by τ in equality (3.128), and for almost all $\tau > 0$ we arrive to the formula

$$MF[f](\tau) = \frac{d}{d\tau} \int_0^\infty P(\tau, y) f(y) dy. \tag{3.129}$$

Turning to the Parseval equality (3.109), observe that it is true for all functions $f(x) \in L_{0,2}(\mathbf{R}_+)$ and corresponding Mehler-Fock transforms $MF[f](\tau) \in L_2\left(\mathbf{R}_+; \frac{4}{\pi^2} t \tanh(\pi t/2)\right)$ by continuity of norms from the relation

$$||f_n||_{L_{0,2}(\mathbf{R}_+)} = ||MF[f_n]||_{L_2\left(\mathbf{R}_+; \frac{4}{\pi^2} t \tanh(\pi t/2)\right)}. \tag{3.130}$$

Hence one can write the Parseval equality (3.109) taking $g(y) = 1,\ 0 < y \le x,\ g(y) = 0, y > x$. Then

$$\int_0^x \frac{f(y)}{y}dy = \frac{2}{\pi}\int_0^\infty \tau \tanh\left(\frac{\pi\tau}{2}\right)\int_0^x P_{-1/2+i\tau/2}(2u^2+1)du MF[f](\tau)d\tau. \quad (3.131)$$

Denoting by

$$\hat{P}(x,\tau) = \int_0^x P_{-1/2+i\tau/2}(2u^2+1)du, \quad (3.132)$$

after differentiation for almost all $x > 0$ we obtain the reciprocal formula for the inverse Mehler-Fock transform as

$$f(x) = \frac{2x}{\pi}\frac{d}{dx}\int_0^\infty \tau \tanh\left(\frac{\pi\tau}{2}\right)\hat{P}(x,\tau)MF[f](\tau)d\tau. \quad (3.133)$$

Similarly to results of Section 2.3 prove that formula (3.104) takes place, in other words the Mehler-Fock transform $MF[f](\tau)$ is the limit in mean square by the norm of space $L_2\left(\mathbf{R}_+; \frac{4}{\pi^2}\tau\tanh(\pi\tau/2)\right)$ of the integral

$$\frac{\pi}{2}\int_{1/N}^N P_{-1/2+i\tau}(2y^2+1)f(y)dy,$$

where $f(x)$ is an arbitrary function from the space $L_{0,2}(\mathbf{R}_+)$. Indeed, in the equality

$$\begin{aligned} MF[f_N](\tau) &= \frac{\pi}{2}\frac{d}{d\tau}\int_0^\infty P(\tau,y)f_N(y)dy \\ &= \frac{\pi}{2}\frac{d}{d\tau}\int_{1/N}^N P(\tau,y)f(y)dy \end{aligned} \quad (3.134)$$

differentiate through due to uniformly convergence of the integral. Thus we deduce the formula

$$MF[f_N](\tau) = \frac{\pi}{2}\int_{1/N}^N P_{-1/2+i\tau/2}(2y^2+1)f(y)dy. \quad (3.135)$$

If $MF[f]$ defined by formula (3.129) then the Parseval equality (3.109) gives

$$\begin{aligned} ||MF[f] - MF[f_N]||^2_{L_2\left(\mathbf{R}_+;\frac{4}{\pi^2}\tau\tanh(\pi\tau/2)\right)} &= ||f - f_N||^2_{L_{0,2}(\mathbf{R}_+)} \\ &= \int_{y\notin[1/N,N]} |f(y)|^2\frac{dy}{y} \to 0,\ N\to\infty, \end{aligned} \quad (3.136)$$

which means that $MF[f_N] \to MF[f]$ by the norm of the space $L_2\left(\mathbf{R}_+; \frac{4}{\pi^2}\tau\tanh(\pi\tau/2)\right)$. Similarly we prove the convergence in mean of the sequence $\{f_N\}$ to f by the norm of $L_{0,2}$, if

$$f_N(x) = \frac{2x}{\pi}\int_0^N \tau\tanh\left(\frac{\pi\tau}{2}\right)P_{-1/2+i\tau/2}(2x^2+1)MF[f](\tau)d\tau. \quad (3.137)$$

Thus we proved

Theorem 3.9. *The operator of the Mehler-Fock transform given by formula* (3.129) *maps the space* $L_{0,2}(\mathbf{R}_+)$ *onto the space* $L_2\left(\mathbf{R}_+; \frac{4}{\pi^2}\tau\tanh(\pi\tau/2)\right)$ *and its inversion is given by formula* (3.133). *These operators are the limits in mean by respective norm of the Hilbert weighted spaces of integrals* (3.135), (3.137).

3.5 An index-convolution transform related to the Mehler-Fock integral

We now demonstrate the composition method of the integral transform constructions, introducing the index-convolution transform like (2.150) related to the Mehler-Fock integral (3.1). Using representation (3.2) one can consider the following kernel

$$\frac{\pi}{2x\cosh(\pi\tau/2)}P_{-1/2+i\tau/2}\left(2\left(\frac{y}{x}\right)^2+1\right)=\int_0^\infty J_0(yt)K_{i\tau}(xt)dt,\ x,y>0, \tag{3.138}$$

and introduce the index-convolution transform of type

$$MF[f](\tau,x)\equiv g(\tau,x)=\frac{\pi}{2x\cosh(\pi\tau/2)}\int_0^\infty P_{-1/2+i\tau/2}\left(2\left(\frac{y}{x}\right)^2+1\right)f(y)dy. \tag{3.139}$$

Here $(\tau,x)\in\mathbf{R}_+\times\mathbf{R}_+$ and $f(y)$ is an arbitrary measurable complex-valued function. We shall follow to results of Section 2.5 to invert transform (3.139), and first let us establish its connection with the index-convolution Kontorovich-Lebedev transform (2.150).

Theorem 3.10. *If $f\in L_{\nu,1}(\mathbf{R}_+)$, $1/2<\nu<1$, then operator* (3.139) *can be represented as*

$$MF[f](\tau,x)=KL[(J_0f)](\tau,x), \tag{3.140}$$

where $KL[f]$ is the index-convolution Kontorovich-Lebedev transform (2.150) *and operator $(J_\mu f)$ is the modified Hankel transform like* (1.225) *defined by formula*

$$(J_\mu f)(x)=\int_0^\infty J_\mu(xt)f(t)dt. \tag{3.141}$$

Proof. Indeed, according to equality (3.138) the proof of this fact is to change the order of integration in iterated integral

$$g(\tau,x)=\int_0^\infty f(y)\int_0^\infty J_0(yt)K_{i\tau}(xt)dtdy \tag{3.142}$$

for almost all $x,\tau>0$. By using estimate (1.100) for the Macdonald function and the weighted Hölder inequality (1.21) in the inner integral (3.142) it can be estimated as

$$\int_0^\infty |J_0(yt)K_{i\tau}(xt)|dt\le e^{-\delta\tau}\left(\int_0^\infty t^{q(1-\nu)-1}|J_0(yt)|^q dt\right)^{1/q}$$
$$\times\left(\int_0^\infty t^{p\nu-1}|K_0(xt\cos\delta)|^p dt\right)^{1/p}, \tag{3.143}$$

where $\delta \in (0, \pi/2)$, $p^{-1} + q^{-1} = 1$. The asymptotic behavior of the Bessel functions gives the convergence of integrals at the right-hand side of (3.143), namely

$$\left(\int_0^\infty t^{q(1-\nu)-1}|J_0(ty)|^q dt\right)^{1/q} = y^{\nu-1}\left(\int_0^\infty t^{q(1-\nu)-1}|J_0(t)|^q dt\right)^{1/q} < \infty, \tag{3.144}$$

$$\left(\int_0^\infty t^{p\nu-1}|K_0(xt\cos\delta)|^p dt\right)^{1/p} < \infty, \tag{3.145}$$

for each $x, y > 0$ when $1/2 < \nu < 1$ as is evident from formulae (1.92)-(1.93), (1.96)-(1.97). Therefore, the Fubini theorem immediately gives the desired composition (3.140) in view of the finiteness of the norm $||f||_{\nu,1}$. Theorem 3.10 is proved. •

Thus we can apply now Lemma 2.7 to estimate the norm of operator $MF[f](\tau, x)$ in the space $L_{\nu,p}(\mathbf{R}_+ \times \mathbf{R}_+)$, $p \geq 1$ normed by formula (2.152).

Theorem 3.11. *Let $f(x)$ be from the space $L_{\nu,1}(\mathbf{R}_+)$ with $1/2 < \nu < 1$. Then the operator $MF[f]$ given by formula* (3.139) *is bounded from the space $L_{\nu,1}(\mathbf{R}_+)$ into the space $L_{\nu,p}(\mathbf{R}_+ \times \mathbf{R}_+)$, $p \geq 1$.*

Proof. At first with the aid of the generalized Minkowski inequality (1.10) one can show that the Hankel transform $(J_0 f)(x)$ from composition (3.140) belongs to the space $L_{1-\nu,1}(\mathbf{R}_+)$ under condition $f \in L_{\nu,1}(\mathbf{R}_+)$, $1/2 < \nu < 1$. Actually, we have

$$||(J_0 f)(x)||_{1-\nu,1} = \int_0^\infty x^{-\nu}\left|\int_0^\infty J_0(xy)f(y)dy\right| dx$$
$$\leq \int_0^\infty |f(y)|dy \int_0^\infty x^{-\nu}|J_0(xy)|dx. \tag{3.146}$$

The simple change of variable $xy = t$ reduces the integral with the Bessel function to a positive constant under $1/2 < \nu < 1$ and we obtain finally from (3.146) that

$$||(J_0 f)(x)||_{1-\nu,1} \leq ||f||_{\nu,1}\int_0^\infty x^{-\nu}|J_0(x)|dx = C_\nu ||f||_{\nu,1}. \tag{3.147}$$

Consequently, one can appeal Lemma 2.7 and invoking with composition (3.140) to lead to the expected result. Theorem 3.11 is completely proved. •

Thus developing ideas of Section 2.5 one can arrive to the inversion of the index-convolution Mehler-Fock operator (3.139) by means of operator (2.156). However, we need to assume some additional conditions concerning the existence of the inverse Hankel transform. Namely, due to Titchmarsh [1] from formula (3.141) it follows formally that

$$f(x) = \int_0^\infty xtJ_\mu(xt)(J_\mu f)(t)dt. \tag{3.148}$$

Letting here $\mu = 0$ from the above discussions we have that $(J_0 f)(x) \in L_{1-\nu,1}(\mathbf{R}_+)$, $1/2 < \nu < 1$, if $f(x) \in L_{\nu,1}(\mathbf{R}_+)$. Let us spread the range of parameter ν up to $1/2 < |\nu| < 1$. Hence if $(J_0 f)(x) \in L_{1-\nu,1}(\mathbf{R}_+)$, where $-1 < \nu < -1/2$, then the operator at the right-hand side (3.148) exists and maps the space $L_{1-\nu,1}$ into the space $L_{\nu,1}$ for $\nu \in (-1,-1/2)$. Precisely, we have the estimate from (3.148) similar above

$$||f||_{\nu,1} \leq ||(J_0 f)||_{1-\nu,1} \int_0^\infty x^\nu |J_0(x)| dx < \infty, \tag{3.149}$$

if as is evident $-1 < \nu < -1/2$. Due to formula 2.16.21.1 in Prudnikov et al. [1] we use now the value of the following integral (see also (1.101))

$$\int_0^\infty t^2 J_0(xt) K_{i\tau}(yt) dt = \frac{2}{y^3} \Gamma\left(\frac{3+i\tau}{2}\right) \Gamma\left(\frac{3-i\tau}{2}\right)$$

$$\times {}_2F_1\left(\frac{3+i\tau}{2}, \frac{3-i\tau}{2}; 1; -\frac{x^2}{y^2}\right). \tag{3.150}$$

This formula shall contribute to the kernel of the inverse index-convolution transform related to (3.139). Observe that composition (3.140) enables to apply Theorem 2.10 and to deduce the following relation

$$(J_0 f)(x) = \text{l.i.m.}_{\varepsilon\to 0+} \frac{2x \sin\varepsilon}{\pi^2} \int_0^\infty \int_0^\infty y \cosh((\pi-\varepsilon)\tau)$$

$$\times K_{i\tau}(xy) g(\tau, y) d\tau dy, \tag{3.151}$$

where $g(\tau, y) = MF[f](\tau, y)$ and the limit is meant by $L_{1-\nu,1}$-norm with $1/2 < \nu < 1$. Meanwhile, the integral at the right-hand side of (3.151) belongs to the space $L_{1-\nu,1}(\mathbf{R}_+)$ with $-1 < \nu < -1/2$ if we assume that

$$\int_0^\infty \int_0^\infty e^{((\pi-\varepsilon-\delta)\tau)} y^{\nu-1} |g(\tau, y)| d\tau dy < \infty, \tag{3.152}$$

where $\delta \in (0, \pi/2)$, $-1/2 < \nu < -1$. Indeed, we have the estimate

$$\int_0^\infty x^{-\nu} dx \left| \int_0^\infty \int_0^\infty xy \cosh((\pi-\varepsilon)\tau) \right.$$

$$\left. \times K_{i\tau}(xy) g(\tau, y) d\tau dy \right|$$

$$\leq C_\delta \int_0^\infty x^{1-\nu} K_0(x) dx \int_0^\infty \int_0^\infty e^{((\pi-\varepsilon-\delta)\tau)} y^{\nu-1} |g(\tau, y)| d\tau dy < \infty \tag{3.153}$$

under the above conditions. Consequently, if there exists the limit in (3.151) by the $L_{1-\nu,1}$-norm, with $-1 < \nu < -1/2$, then it coincides with $(J_0 f)(x)$ and we can invert through with the aid of the Hankel operator (3.148). Accounting its boundedness by the $L_{\nu,1}$-norm we write

$$f(x) = \text{l.i.m.}_{\varepsilon\to 0+} \frac{2 \sin\varepsilon}{\pi^2} \int_0^\infty xt J_0(xt) dt$$

$$\times \int_0^\infty \int_0^\infty yt \cosh((\pi-\varepsilon)\tau) K_{i\tau}(yt) g(\tau, y) d\tau dy. \tag{3.154}$$

However, one can change the order of integration by Fubini's theorem in the iterated integral (3.154) according to estimate as

$$\int_0^\infty t^2 |J_0(xt)| dt$$

$$\times \int_0^\infty \int_0^\infty y \cosh((\pi - \varepsilon)\tau) |K_{i\tau}(yt) g(\tau, y)| d\tau dy$$

$$\leq C x^{-1-\nu} \left(\int_0^\infty t^{q(1+\nu)-1} |J_0(t)|^q dt \right)^{1/q}$$

$$\times \left(\int_0^\infty t^{p(2-\nu)-1} |K_0(t)|^p dt \right)^{1/p}$$

$$\times \int_0^\infty \int_0^\infty e^{((\pi-\varepsilon-\delta)\tau)} y^{\nu-1} |g(\tau, y)| d\tau dy < \infty, \tag{3.155}$$

where $p^{-1} + q^{-1} = 1$ and $-1 < \nu < -1/2$. Invoking with equality (3.150) we obtain finally from (3.154) that

$$f(x) = \text{l.i.m.}_{\varepsilon \to 0+} \frac{4x \sin \varepsilon}{\pi^2} \Gamma\left(\frac{3+i\tau}{2}\right) \Gamma\left(\frac{3-i\tau}{2}\right)$$

$$\times \int_0^\infty \int_0^\infty \cosh((\pi - \varepsilon)\tau) {}_2F_1\left(\frac{3+i\tau}{2}, \frac{3-i\tau}{2}; 1; -\frac{x^2}{y^2}\right)$$

$$\times \frac{g(\tau, y)}{y^2} d\tau dy. \tag{3.156}$$

Thus we established the following final result.

Theorem 3.12. *Let* $g(\tau, x) = MF[f](\tau, x)$ *and condition* (3.152) *holds. If* $f(x) \in L_{\nu,1}(\mathbf{R}_+), 1/2 < \nu < 1$ *and the Hankel transform* $(J_0 f)(x) \in L_{1-\nu,1}(\mathbf{R}_+), -1 < \nu < -1/2$ *is the limit in* (3.151) *being meant by* $L_{1-\nu,1}$*-norm, then the inversion formula* (3.156) *for the index-convolution transform* (3.139) *is true.*

Chapter 4

Convolution of the Kontorovich-Lebedev Transform

In the preceding two chapters we introduced the Kontorovich-Lebedev and the Mehler-Fock integral transforms by index of the Macdonald and the Legendre functions as the kernels, respectively. We investigated their mapping properties in the weighted L_p-spaces and obtained some composition relations.

This chapter deals with new objects as *convolutions* connected with the index transforms mentioned above. These integral operators are fundamentally different from the considered convolution operators of Mellin's, Laplace's and Fourier's types. It enables us to illustrate various convolution constructions and apply theirs to the investigation of the respective classes of integral equations of the first and second kind. As is shown the demonstrated convolutions of the Kontorovich-Lebedev transform have excellent mapping properties within Lebesgue spaces of measurable functions which were introduced in Chapter 1.

4.1 Definition of the convolution. Useful estimates

The purpose of the present consideration is to study the convolution operator related to the Kontorovich-Lebedev transform (2.1) defined by the following double integral

$$(f*g)(x) = \frac{1}{2x}\int_0^\infty\int_0^\infty \exp\left(-\frac{1}{2}\left[\frac{xu}{y}+\frac{xy}{u}+\frac{yu}{x}\right]\right) f(u)g(y)dudy, \; x>0, \quad (4.1)$$

where $f(x)$ and $g(x)$ are two functions from suitable functional space. This operator was first introduced in Kakichev [1] formally as an example of integral nonstandard convolution. Later this operator was considered in detail by the author in Yakubovich

[5]-[6], Yakubovich and Moshinskii [1], Yakubovich and Luchko [2]-[3] in slightly different form. Moreover, this convolution was generalized for other index transforms and applications to various type of integral equations were obtained. The related questions for the convolution (4.1) in the space of generalized function were considered in Glaeske and Hess [1]. Its analog for the Mehler-Fock transform was studied in Glaeske and Hess [2]-[3]. This class of convolutions essentially completes the double integral type convolutions in terms of the Mellin-Barnes integrals which were investigated in Nguyen Thanh Hai and Yakubovich [1].

This chapter is intended to give the reader the series of results and estimates related to convolution (4.1) in the weighted spaces $L_{\nu,p}$. As a conclusion we extend our understanding of these objects and their applications to integral equations. Some separate examples of integral equations with convolution (4.1) were considered previously in Lebedev [6], Yakubovich [5], Yakubovich and Luchko [2]. These equations involve operator (4.1) as follows

$$(\mathcal{K}f)(x) = \int_0^\infty \mathcal{K}(x,u)f(u)du, \tag{4.2}$$

where we fixed some function $g(y)$ and calculated the kernel $\mathcal{K}(x,u)$ by the integral

$$\mathcal{K}(x,u) = \frac{1}{2x}\int_0^\infty \exp\left(-\frac{1}{2}\left[\frac{xu}{y}+\frac{xy}{u}+\frac{yu}{x}\right]\right)g(y)dy, \ x>0. \tag{4.3}$$

We shall touch these questions below and shall demonstrate interesting examples of integral equations and their solutions.

We start to study mapping properties of convolution (4.1). First observe from definition that convolution (4.1) is symmetrical (commutative)

$$f * g = g * f. \tag{4.4}$$

Second, if for each $x \in \mathbf{R}_+$ $f(x) > 0,\ g(x) > 0$ $(f(x) < 0,\ g(x) < 0)$, then $(f*g)(x) > 0$, and for $f(x) > 0,\ g(x) < 0,\ (f(x) < 0,\ g(x) > 0)$ the inequality $(f * g)(x) < 0$ is justified.

Now one can obtain certain estimates for convolution (4.1) in the Lebesgue L_p-spaces applying in all cases the Fubini Theorem 1.1.

Theorem 4.1. *Let $f(x)$, $g(x)$ be functions from the space $L_{1/2,1}(\mathbf{R}_+)$. Then convolution* (4.1) *exists and satisfies the estimate*

$$|(f*g)(x)| \le \frac{e^{-x}}{2\sqrt{2x}}||f||_{1/2,1}||g||_{1/2,1}. \tag{4.5}$$

Proof. Using the elementary inequalities

$$e^{-x} \le \frac{1}{1+x}, \ x>0, \tag{4.6}$$

$$a^2 + b^2 \geq 2ab \tag{4.7}$$

deduce following relations

$$\begin{aligned}
|(f*g)(x)| &\leq \frac{1}{2x}\int_0^\infty \int_0^\infty \frac{\exp\left(-x\frac{u^2+t^2}{2ut}\right)}{1+\frac{ut}{2x}}|f(t)g(u)|dtdu \\
&\leq \frac{e^{-x}}{2\sqrt{2x}}\int_0^\infty \int_0^\infty \frac{|f(t)|}{\sqrt{t}}\frac{|g(u)|}{\sqrt{u}}dtdu \\
&= \frac{e^{-x}}{2\sqrt{2x}}||f||_{1/2,1}||g||_{1/2,1},
\end{aligned} \tag{4.8}$$

which give the desired result. Theorem 4.1 is proved. •

Theorem 4.2. *Let $f(x)$, $g(x)$ be functions from the weighted space $L_{0,1}(\mathbf{R}_+)$. Then convolution* (4.1) *exists for each $x > 0$, and moreover,*

$$|(f*g)(x)| \leq e^{-x}||f||_{0,1}||g||_{0,1}. \tag{4.9}$$

Proof. By virtue of inequality (4.6) it follows that

$$\begin{aligned}
|(f*g)(x)| &\leq \frac{1}{2x}\int_0^\infty \int_0^\infty \exp\left(-x\frac{u^2+t^2}{2ut}\right)\frac{2x}{2x+ut}|f(t)g(u)|dtdu \\
&\leq e^{-x}\int_0^\infty \frac{|f(t)|}{t}dt \int_0^\infty \frac{|g(u)|}{u}du,
\end{aligned} \tag{4.10}$$

that implies conclusion (4.9) of Theorem 4.2. •

The next results involve L-spaces with power-exponential weights.

Theorem 4.3. *Let as usually $p \geq 1$, $q \geq 1$, $\frac{1}{p}+\frac{1}{q} = 1$. Let $f(x)$ and $g(x)$ be from the weighted space $L\left(\mathbf{R}_+; x^{-1/2}\exp\left(-\frac{x}{2\min(p,q)}\right)\right)$. Then the convolution $(f*g)(x)$ exists for each $x \in \mathbf{R}_+$, and estimate*

$$\begin{aligned}
|(f*g)(x)| &< \sqrt{\frac{\max(p,q)}{8x}}\exp\left(-\frac{x}{2}\left(1+\frac{1}{\max(p,q)}\right)\right) \\
&\times ||f||_{L(\mathbf{R}_+;x^{-1/2}\exp(-x/(2\min(p,q)))}\ ||g||_{L(\mathbf{R}_+;x^{-1/2}\exp(-x/(2\min(p,q)))}
\end{aligned} \tag{4.11}$$

holds.

Proof. According to the definition of the convolution, we have the sequence of equalities

$$(f*g)(x) = \frac{1}{2x}\int_0^\infty \int_0^\infty \exp\left(-\frac{1}{2p}\left[\frac{xt}{u}+\frac{xu}{t}+\frac{ut}{x}\right]\right)$$

$$\times \exp\left(-\frac{1}{2q}\left[\frac{xt}{u}+\frac{xu}{t}+\frac{ut}{x}\right]\right) f(t)g(u)dtdu$$

$$= \frac{1}{2x}\int_0^\infty \int_0^\infty \exp\left(-x\frac{u^2+t^2}{2ut\max(p,q)} - \frac{ut}{2x\max(p,q)}\right)$$

$$\times \exp\left(-\frac{1}{2\min(p,q)}\left[\frac{xt}{u}+\frac{xu}{t}+\frac{ut}{x}\right]\right) f(t)g(u)dtdu. \tag{4.12}$$

Recalling inequalities (4.6)-(4.7) to (4.12) and the inequality

$$\frac{xu}{t}+\frac{xt}{u}+\frac{ut}{x} \geq x+u+t, \tag{4.13}$$

we easily deduce that

$$|(f*g)(x)| \leq \sqrt{\frac{\max(p,q)}{8x}} \exp\left(-x\left[\frac{1}{\max(p,q)}+\frac{1}{2\min(p,q)}\right]\right)$$

$$\times \int_0^\infty \frac{|f(x)|}{\sqrt{t}} \exp\left(-\frac{t}{2\min(p,q)}\right) dt \int_0^\infty \frac{|g(u)|}{\sqrt{u}} \exp\left(-\frac{u}{2\min(p,q)}\right) du, \tag{4.14}$$

and hence relation (4.11) is satisfied. Theorem 4.3 is proved. •

Theorem 4.4. *Let $f(x)$, $g(x)$ belong to the space $L(\mathbf{R}_+; e^{-x/2})$. Then convolution* (4.1) *exists for each $x \in \mathbf{R}_+$ and the estimate*

$$|(f*g)(x)| \leq \frac{e^{-x}}{2x}||f||_{L(\mathbf{R}_+;e^{-x/2})}||g||_{L(\mathbf{R}_+;e^{-x/2})} \tag{4.15}$$

is true.

Proof. The desired result easily follows from the inequality demonstrated above (4.13) applying it to the kernel

$$\exp\left(-\frac{1}{2}\left(\frac{xt}{u}+\frac{xu}{t}+\frac{ut}{x}\right)\right). \tag{4.16}$$

Indeed, it follows that

$$\exp\left(-\frac{1}{2}\left(\frac{xt}{u}+\frac{xu}{t}+\frac{ut}{x}\right)\right) \leq e^{-x/2}e^{-t/2}e^{-u/2}. \tag{4.17}$$

Consequently, the double integral in (4.1) reduced to the product of two integrals with the outer power-exponential coefficient. It gives us inequality (4.15). Theorem 4.4. is proved. •

As is obvious from the above theorems that convolution (4.1) is continuous function of the variable $x \in \mathbf{R}_+$ and exponentially decreases at infinity.

Theorem 4.5. *Let for $\Re\alpha < 1$ the function $x^\alpha g(x)$ be bounded on the half-axis $(0, \infty)$, and $f(x)$ be a function from $L_{\alpha,1}(\mathbf{R}_+)$. Then the convolution $(f * g)(x)$ exists for each $x > 0$ and*

$$|(f * g)(x)| < M 2^{-\alpha}\Gamma(1-\alpha)x^{-\alpha}e^{-x}||f||_{L_{\alpha,1}(\mathbf{R}_+)}, \tag{4.18}$$

where $M > 0$ is a constant.

Proof. Indeed, observe that

$$\begin{aligned} |(f * g)(x)| <& \frac{1}{2x}\int_0^\infty \int_0^\infty \exp\left(-\frac{1}{2}\left[\frac{xu}{t} + \frac{ut}{x} + \frac{tx}{u}\right]\right)|f(t)u^\alpha g(u)|\frac{dtdu}{u^\alpha} \\ &< \frac{M}{2x}\int_0^\infty |f(t)|dt \int_0^\infty \exp\left(-x\frac{u^2+t^2}{2ut}\right)\exp\left(-\frac{ut}{2x}\right)u^{-\alpha}du \\ &< \frac{M\Gamma(1-\alpha)}{2x}e^{-x}\int_0^\infty |f(t)|\left(\frac{t}{2x}\right)^{\alpha-1}dt, \end{aligned} \tag{4.19}$$

which leads us to estimate (4.18). Theorem 4.5 is proved. •

Theorem 4.6. *Let the function $x^\alpha g(x)$ be bounded for $\Re\alpha < 1$ on the interval $(0, \infty)$, and $f(x) \in L_2(\mathbf{R}_+)$. Then convolution $(f * g)(x)$ (4.1) exists for each $x \in (0, \infty)$ and the following estimate is true*

$$|(f * g)(x)| < M_1 x^{\alpha-2}\exp\left(-\frac{x}{p}\right)||f||_{L_2(\mathbf{R}_+)}, \tag{4.20}$$

where $p > 1$ is an arbitrary number, $M_1 > 0$ is a constant.

Proof. Applying the Hölder inequality (1.8) to convolution (4.1), we obtain the representation

$$\begin{aligned} |(f * g)(x)| < \frac{1}{2x}\left(\int_0^\infty \left|\int_0^\infty \exp\left(-\frac{1}{2}\left(\frac{xu}{t} + \frac{ut}{x} + \frac{xt}{u}\right)\right)g(u)du\right|^2 dt\right)^{1/2} \\ \times\left(\int_0^\infty |f(t)|^2 dt\right)^{1/2}. \end{aligned} \tag{4.21}$$

Treat now the inner integral in (4.21). Taking the parameters $p > 1,\ q > 1$, for which $\frac{1}{p} + \frac{1}{q} = 1$ and one can reduce it to the form

$$\begin{aligned} \int_0^\infty \exp\left(-\frac{1}{2}\left[\frac{xu}{t} + \frac{ut}{x} + \frac{xt}{u}\right]\right)g(u)du = \int_0^\infty \exp\left(-\frac{1}{2p}\left[\frac{xu}{t} + \frac{ut}{x} + \frac{xt}{u}\right]\right) \\ \times\exp\left(-\frac{1}{2q}\left[\frac{xu}{t} + \frac{ut}{x} + \frac{xt}{u}\right]\right)g(u)du. \end{aligned} \tag{4.22}$$

Further, the simple inequality (4.7) allows us to obtain that

$$\left|\int_0^\infty \exp\left(-\frac{1}{2}\left[\frac{xu}{t}+\frac{ut}{x}+\frac{xt}{u}\right]\right) g(u)du\right| < \exp\left(-\frac{x}{p}\right)\exp\left(-\frac{t}{q}\right)$$

$$\times \int_0^\infty \exp\left(-\frac{u}{2}\left(\frac{t}{px}+\frac{x}{qt}\right)\right)|g(u)|du. \tag{4.23}$$

According to the condition of the present theorem the function $x^\alpha g(x)$ is uniformly bounded on $\mathbf{R}_+$, namely $x^\alpha|g(x)| < C$, where $C > 0$ is some constant. Thus, we arrive to the relations

$$\int_0^\infty \exp\left(-\frac{u}{2}\left(\frac{t}{px}+\frac{x}{qt}\right)\right)|g(u)|du$$

$$< C\int_0^\infty \exp\left(-\frac{u}{2}\left(\frac{t}{px}+\frac{x}{qt}\right)\right)u^{-\alpha}du = C\Gamma(1-\alpha)2^{1-\alpha}\left(\frac{t}{px}+\frac{x}{qt}\right)^{\alpha-1}$$

$$= C_1\Gamma(1-\alpha)x^{1-\alpha}\left(\frac{t}{qt^2+px^2}\right)^{1-\alpha} < C_2\Gamma(1-\alpha)x^{\alpha-1}t^{1-\alpha}, \tag{4.24}$$

where C_1, C_2 are constants. Returning to inequality (4.22) in view of (4.23), (4.24), we obtain estimate (4.21). Theorem 4.6 is proved. •

We attract our attention now to the next result that is formulated in the theorem below and estimates the $L_{\nu,p}$-norm (1.19) of convolution (4.1).

Theorem 4.7. *Let $f(x)$, $g(x) \in L_p(\mathbf{R}_+)$, where $1 \le p \le \infty$. Then convolution (4.1) of the Kontorovich-Lebedev transform exists and belongs to $L_{\nu,q}(\mathbf{R}_+)$, $q = p/(p-1)$, $\nu > 1/p$. Moreover, under these conditions*

$$||(f*g)(x)||_{\nu,q} \le C||g||_p||f||_p, \tag{4.25}$$

where $C > 0$ is an absolute constant.

Proof. Making use the generalized Minkowski inequality (1.10), we have

$$||(f*g)(x)||_{\nu,q} = \frac{1}{2}\left(\int_0^\infty x^{q(\nu-1)-1}dx\left|\int_0^\infty\int_0^\infty \exp\left(-\frac{1}{2}\left[\frac{xu}{t}+\frac{xt}{u}+\frac{ut}{x}\right]\right)\right.\right.$$

$$\left.\left.\times f(u)g(t)dudt\right|^q\right)^{1/q} \le \int_0^\infty\int_0^\infty |f(u)g(t)|$$

$$\times\frac{1}{2}\left(\int_0^\infty x^{q(\nu-1)-1}\exp\left(-\frac{q}{2}\left(\frac{xu}{t}+\frac{xt}{u}+\frac{ut}{x}\right)\right)dx\right)^{1/q}dudt. \tag{4.26}$$

The integral by x can be calculated invoking with formula 2.3.16.1 in Prudnikov et al. [1] which shows us that

$$\frac{1}{2}\left(\int_0^\infty x^{q(\nu-1)-1}\exp\left(-\frac{q}{2}\left(\frac{xu}{t}+\frac{xt}{u}+\frac{ut}{x}\right)\right)dx\right)^{1/q}$$

$$= \left(\frac{ut}{\sqrt{u^2+t^2}}\right)^{\nu-1} K^{1/q}_{q(\nu-1)}\left(q\sqrt{u^2+t^2}\right). \tag{4.27}$$

Hence the right-hand side of the inequality in (4.26) reduces to the double integral and the estimate becomes as

$$||(f*g)||_{\nu,q} \leq \int_0^\infty \int_0^\infty \left(\frac{ut}{\sqrt{u^2+t^2}}\right)^{\nu-1} K^{1/q}_{q(\nu-1)}\left(q\sqrt{u^2+t^2}\right)|f(u)g(t)|dudt. \tag{4.28}$$

To continue inequality (4.28) apply the Hölder inequality (1.8) meaning that it involves multidimensional integrals too. In our case for the double integral we obtain that

$$||(f*g)||_{\nu,q} \leq \left(\int_0^\infty \int_0^\infty \left(\frac{ut}{\sqrt{u^2+t^2}}\right)^{q(\nu-1)} K_{q(\nu-1)}\left(q\sqrt{u^2+t^2}\right) dudt\right)^{1/q}$$

$$\times \left(\int_0^\infty |f(u)|^p du\right)^{1/p} \left(\int_0^\infty |g(t)|^p dt\right)^{1/p}. \tag{4.29}$$

To establish inequality (4.25) one may prove the convergence of the double integral in (4.29) with the Macdonald function $K_{q(\nu-1)}\left(q\sqrt{u^2+t^2}\right)$. To show it appeal to the polar coordinates $u = r\cos\varphi,\ t = r\sin\varphi,\ r > 0, \varphi \in (0, \pi/2)$. Thus we find

$$\int_0^\infty \int_0^\infty \left(\frac{ut}{\sqrt{u^2+t^2}}\right)^{q(\nu-1)} K_{q(\nu-1)}\left(q\sqrt{u^2+t^2}\right) dudt$$

$$= 2^{q(1-\nu)} \int_0^{\pi/2} \sin^{q(\nu-1)}(2\varphi)d\varphi \int_0^\infty r^{q(\nu-1)+1} K_{q(\nu-1)}(qr)dr. \tag{4.30}$$

Taking into account the asymptotic behavior of the Macdonald function $K_\mu(x)$ by formulae (1.96)-(1.97) it is not difficult to conclude that the integral by r is convergent for any ν whereas the integral by φ as it is easily seen is convergent only for $\nu > 1/p,\ p = q/(q-1)$. Hence denoting integral (4.30) as C we arrive to inequality (4.25). This completes the proof of Theorem 4.7. •

It is naturally to see now that convolution (4.1) belongs to the conjugate space $L_q(\mathbf{R}_+)$ if we put in inequality (4.25) $\nu = 1/q,\ q < p$ that means $1 \leq q < 2,\ p \geq 2$ or $1 \leq q \leq 2,\ p > 2$.

Corollary 4.1. *The convolution operator* (4.2) *with kernel* $\mathcal{K}(x,u)$ *in* (4.3) *is a bounded one from the space* $L_p(\mathbf{R}_+),\ p \geq 1$ *into the space* $L_{\nu,q}(\mathbf{R}_+),\ q = p/(p-1),\ \nu > 1/p$ *under condition* $g(x) \in L_p(\mathbf{R}_+)$, *where* $g(x)$ *is a characteristic function of the kernel* (4.3).

Using the Hölder inequality let us estimate now the kernel $\mathcal{K}(x,u)$ of operator (4.2) provided that $g(x) \in L_p(\mathbf{R}_+)$. Indeed, we have

$$|\mathcal{K}(x,u)| \leq \frac{1}{2x}\int_0^\infty \exp\left(-\frac{1}{2}\left(\frac{xu}{y}+\frac{xy}{u}+\frac{uy}{x}\right)\right)|g(y)|dy$$

$$\leq \frac{1}{2x}\left(\int_0^\infty \exp\left(-\frac{q}{2}\left(\frac{xu}{y}+\frac{xy}{u}+\frac{uy}{x}\right)\right)dy\right)^{1/q} ||g||_p. \tag{4.31}$$

Recalling again formula 2.3.16.1 in Prudnikov et al. [1] calculate the integral with exponent and deduce the following final estimate

$$|\mathcal{K}(x,u)| \leq ||g||_p \left(\frac{u}{\sqrt{u^2+x^2}}\right)^{1/q} K_1^{1/q}(q\sqrt{u^2+x^2})x^{-1/p}, \tag{4.32}$$

where $K_1(z)$ is the Macdonald function of the index 1. As it is not difficult to see this kernel has immovable singularity at the point $x = u \to 0+$. Precisely speaking, we obtain in this case that $\mathcal{K}(x,x) = O(1/x), x \to 0+$.

4.2 The factorization property. Parseval's type equality

In this section we show the connection between the convolution introduced above and the Kontorovich-Lebedev transform (2.1) by means of his action through the operator of convolution (4.1). Thus one can prove the following theorem.

Theorem 4.8. *Let $f(x)$, $g(x) \in L_p(\mathbf{R}_+)$, for $1/p < \nu < 1$, $p > 1$. Then the Kontorovich-Lebedev transform* (2.1) *of convolution* (4.1) *$(f*g)(x)$ for functions $f(x)$ and $g(x)$ exists and is equal to the product of the Kontorovich-Lebedev transforms for these functions, namely the factorization property*

$$K_{i\tau}[(f*g)] = K_{i\tau}[f]K_{i\tau}[g] \tag{4.33}$$

takes place. Furthermore, the following Parseval's type equality holds

$$(f*g)(x) = \frac{2}{\pi^2}\int_0^\infty \tau\sinh(\pi\tau)\frac{K_{i\tau}(x)}{x}K_{i\tau}[f]K_{i\tau}[g]d\tau, \tag{4.34}$$

for any $x > 0$ and integral (4.34) *is absolutely convergent.*

Proof. The existence of the Kontorovich-Lebedev transform (2.1) of convolution (4.1) follows from Theorem 4.7, because according to (2.13) $K_{i\tau}[(f*g)] \in KL(L_{\nu,q})$. Hence apply through the Kontorovich-Lebedev operator to convolution (4.1) and obtain the iterated integral as

$$K_{i\tau}[(f*g)] = \int_0^\infty \frac{K_{i\tau}(y)}{y}\int_0^\infty\int_0^\infty \exp\left(-\frac{1}{2}\left(\frac{yt}{u}+\frac{yu}{t}+\frac{ut}{y}\right)\right)$$

$$\times f(u)g(t)dudtdy. \tag{4.35}$$

The inner integral by y is calculated by means of the mentioned Macdonald formula (1.103). Hence, change the order of integration in which we perform, motivating it by inequality (1.100) and estimate like (2.4). After using the Macdonald formula (1.103) we deduce (4.33). Furthermore, taking integral representation 2.16.56.1 in Prudnikov et al. [2] for the exponential kernel (4.16), namely the formula

$$\exp\left(-\frac{1}{2}\left(\frac{xy}{u}+\frac{xu}{y}+\frac{yu}{x}\right)\right)=\frac{4}{\pi^2}\int_0^\infty \tau\sinh(\pi\tau)K_{i\tau}(x)K_{i\tau}(y)K_{i\tau}(u)d\tau, \qquad (4.36)$$

substitute it in (4.1). To change the order of integration use the estimate that follows in view of (1.100), precisely we write

$$\int_0^\infty \tau\sinh(\pi\tau)|K_{i\tau}(x)K_{i\tau}(y)K_{i\tau}(u)|d\tau$$

$$\leq K_0(x\cos\delta_1)K_0(y\cos\delta_2)K_0(u\cos\delta_3)\int_0^\infty \tau\sinh(\pi\tau)e^{-\tau(\delta_1+\delta_2+\delta_3)}d\tau<+\infty, \qquad (4.37)$$

where $\delta_i\in[0,\pi/2),\ i=1,2,3$ and as is obvious one can choose these parameters to satisfy the convergence of integral (4.37). Further, apply the Hölder inequality, which gives us the relations

$$|(f*g)(x)|\leq\frac{2}{\pi^2}\frac{K_0(x\cos\delta_1)}{x}\int_0^\infty \tau\sinh(\pi\tau)e^{-\tau(\delta_1+\delta_2+\delta_3)}d\tau$$

$$\times||f||_p||g||_p\left(\int_0^\infty K_0^q(y\cos\delta_2)dy\right)^{1/q}\left(\int_0^\infty K_0^q(u\cos\delta_3)du\right)^{1/q}. \qquad (4.38)$$

Hence we verified in changing the order of integration and immediately establish equality (4.34). The Theorem 4.8 is proved. •

Let us illustrate now special weighted suitable space for convolution (4.1) $L^\alpha\equiv L\left(\mathbf{R}_+;K_\alpha(x)\right)$, $\alpha\geq 0$ with norm (1.18) putting there $\rho(t)=K_\alpha(t),\ p=1$. We draw a parallel and slightly modify results from Yakubovich and Luchko [2]. One can prove that this space of absolutely integrable functions on $\mathbf{R}_+$ with the weight $K_\alpha(t)$ as the Macdonald function of index α form normed ring or a Banach algebra with the norm

$$||f||_{L^\alpha}=\int_0^\infty K_\alpha(t)|f(t)|dt<+\infty. \qquad (4.39)$$

First from the asymptotic behavior of the Macdonald function observe the following evident embedding

$$L^{\alpha_1}\subseteq L^{\alpha_2},\ \text{iff}\ \alpha_1\geq\alpha_2. \qquad (4.40)$$

Theorem 4.9. *Let $f(x)$, $g(x)$ be from the space $L(\mathbf{R}_+;K_\alpha(x))$. Then convolution (4.1) exists and belongs to the class $L(\mathbf{R}_+;K_\alpha(x))$. In addition,*

$$||f*g||_{L^\alpha}\leq||f||_{L^\alpha}||g||_{L^\alpha}. \qquad (4.41)$$

Proof. Equality (4.39) of the norm L^α leads us to the following inequality

$$||f * g||_{L^\alpha} = \int_0^\infty K_\alpha(x)|(f * g)(x)|dx$$

$$\leq \frac{1}{2}\int_0^\infty \frac{K_\alpha(x)}{x}\int_0^\infty\int_0^\infty \exp\left(-\frac{1}{2}\left(\frac{xy}{u}+\frac{uy}{x}+\frac{xu}{y}\right)\right)|f(u)g(y)|dydudx. \tag{4.42}$$

One can perform to change the order of integration by Fubini's theorem in the last iterated integral, using the Macdonald formula (1.103), because that integral is convergent and we obtain

$$\frac{1}{2}\int_0^\infty \frac{K_\alpha(x)}{x}\int_0^\infty\int_0^\infty \exp\left(-\frac{1}{2}\left(\frac{xy}{u}+\frac{uy}{x}+\frac{xu}{y}\right)\right)$$

$$\times|f(u)g(y)|dydudx = \frac{1}{2}\int_0^\infty\int_0^\infty |f(u)g(y)|$$

$$\times\int_0^\infty K_\alpha(x)\exp\left(-\frac{1}{2}\left(\frac{xy}{u}+\frac{uy}{x}+\frac{xu}{y}\right)\right)\frac{dx}{x}dudy$$

$$= \int_0^\infty K_\alpha(u)|f(u)|du\int_0^\infty K_\alpha(y)|g(y)|dy = ||f||_{L^\alpha}||g||_{L^\alpha}. \tag{4.43}$$

This completes the proof of Theorem 4.9. •

As is known from the theory of Lebesgue's integral, the property of the integrability of convolution (4.1) with the positive weight $K_\alpha(x)$ shows that for almost all $x > 0$ it takes finite values. Let us formulate now the theorem that provides the validity of formula (4.33) in the space L^α.

Theorem 4.10. *Let $f(x)$, $g(x)$ be from the space L^α. Then the Kontorovich-Lebedev transform* (2.1) *of the convolution $(f*g)(x)$ exists and is equal to the product of the Kontorovich-Lebedev transforms of convolution functions $f(x)$ and $g(x)$, that is, formula* (4.33) *is justified.*

Proof. The existence of the Kontorovich-Lebedev transform for convolution follows from previous theorem and the estimate

$$|K_{i\tau}[f * g]| \leq \int_0^\infty K_0(y)|(f * g)(y)|dy$$

$$< \int_0^\infty K_0(u)|f(u)|du\int_0^\infty K_0(y)|g(y)|dy < \infty, \tag{4.44}$$

owing to embedding (4.40). This allows us to change the order of integration in the corresponding iterated integral and use the Macdonald formula (1.103). Theorem 4.10 is proved. •

Our purpose in this section is to consider also the subspace of the space L^α, which we denote as $L^\alpha_\beta \equiv L_1(\mathbf{R}_+; K_\alpha(\beta x))$, $\alpha \geq 0$, $0 < \beta \leq 1$. The embedding

$$L^\alpha_\beta \subseteq L^\alpha \tag{4.45}$$

evidently follows from asymptotics (1.96)-(1.97) of the Macdonald function and $L^\alpha_1 \equiv L^\alpha$. The next theorem is true.

Theorem 4.11. *Let $f(x)$, $g(x)$ be from the space $L^\alpha_{\cos\delta}$, $\pi/3 < \delta < \pi/2$. Then the Parseval equality* (4.34) *holds.*

Proof. Indeed, by virtue of the estimate

$$|(f * g)(x)| \leq \frac{2}{\pi^2}\frac{K_0(x\cos\delta)}{x}\int_0^\infty \tau\sinh(\pi\tau)e^{-3\tau\delta}d\tau$$
$$\times \int_0^\infty K_0(u\cos\delta)|f(u)|du\int_0^\infty K_0(y\cos\delta)|g(y)|dy, \tag{4.46}$$

it is easily follows that integrals by u and y are finite, because the same embedding (4.40) for the set of spaces L^α_β is true when β is a fixed number. Hence according to the condition on δ the integral by τ is convergent. Therefore we apply formula (4.36), substitute it in convolution (4.1), change the order of integration and invoking with (2.1) arrive to the Parseval equality (4.34). This completes the proof of Theorem 4.11. •

Note here that the range of the parameter ν in Macdonald formula (1.103) is an arbitrary complex number. Therefore, by virtue of the condition that $f(x)$ is an element of the space L^α, factorization equality (4.33) can be extended on the Kontorovich-Lebedev transform (2.84) $K_s[f]$ with the index $s = \mu + i\tau$ from the strip $|\mu| \leq \alpha$. Indeed, due to the asymptotic behavior of the Macdonald function, the corresponding integral remains absolutely and uniformly convergent and gives an analytic function in the mentioned strip.

Considering the question of one-to-one correspondence of the function from the space L^α and its Kontorovich-Lebedev's transform $K_s[f]$ of complex index s we obtain the following theorem.

Theorem 4.12. *If the Kontorovich-Lebedev transform* (2.84) $K_s[f]$, $s = \mu + i\tau$ *of a function $f(x)$ from L^α, $\alpha \geq |\mu|$ is identically zero, then $f(x)$ is equal to zero almost everywhere on $\mathbf{R}_+$.*

Proof. Representation (1.99) of the Macdonald function $K_s(x)$ can be written in the form

$$K_s(x) = \frac{1}{2}\int_{-\infty}^\infty e^{-x\cosh u - su}du. \tag{4.47}$$

Since $f(x) \in L(\mathbf{R}_+; K_\alpha(x))$, then according to the estimate

$$\left|\int_0^\infty K_s(t)f(t)dt\right| = \frac{1}{2}\left|\int_0^\infty f(t)dt\int_{-\infty}^\infty e^{-t\cosh u - su}du\right|$$

$$\leq \int_0^\infty |f(t)| K_\mu(t) dt < \infty, \tag{4.48}$$

when $\alpha \geq |\mu|$ (see asymptotic of the Macdonald function (1.96)) . Thus using the Fubini theorem , we have the following composition representation

$$K_s[f] = \sqrt{\frac{\pi}{2}} \left[F\, e^{-\mu u} \left[Lf \right] (\cosh u) \right] (-\tau), \tag{4.49}$$

where $[Ff](-\tau)$ is the Fourier transform (1.191) and the symbol $[Lf](\cosh u)$ means the Laplace transform (1.215) which is calculated at the point $\cosh u$. By the same way it is easy to show that the function $e^{-\mu u}[Lf](\cosh u) \in L_1(\mathbf{R})$. Indeed, we find that

$$\int_{-\infty}^\infty e^{-\mu u} \left| \int_0^\infty f(t) e^{-t\cosh u} dt \right| du$$
$$\leq 2 \int_0^\infty |f(t)| K_\mu(t) dt < \infty, \; |\mu| \leq \alpha. \tag{4.50}$$

Hence one can conclude that the Kontorovich-Lebedev transform $K_s[f]$ is an analytic function at the strip $|\mu| < \alpha$. Moreover, according to the well-known property of the Fourier transform of an absolutely integrable function (see for example, in Titchmarsh [1]) the equality

$$K_s[f] \equiv 0, \; s = \mu + i\tau \tag{4.51}$$

implies that for almost every $u \in \mathbf{R}$

$$\int_0^\infty e^{-x \cosh u} f(x) dx = 0. \tag{4.52}$$

It is easily to note from properties of the space $L(\mathbf{R}_+; K_\alpha(x))$, that for $u \geq u_0 > 0$, integral (4.52) converges absolutely and uniformly, that is, it defines continuous function. Hence, it follows that equality (4.52) is an identity for $u \geq u_0 > 0$. Further, there is $\varepsilon > 0$, such that $v = \cosh(u) - 1 - \varepsilon > 0$. Then equality (4.52) takes the form

$$\int_0^\infty e^{-vx} f(x) e^{-(1+\varepsilon)x} dx \equiv 0. \tag{4.53}$$

Meanwhile, from identity (4.53) and from the condition of the theorem, it follows that the Laplace transform of the absolutely integrable function $f(x)e^{-(1+\varepsilon)x}$ is identically zero on some closed interval $0 < a \leq v \leq b$ and, moreover, on the right half-plane $\Re z \geq a$ (for $v = z$). Therefore, as it is clear from the uniqueness theorem for analytic functions, we find that

$$\int_0^\infty e^{-zx} f(x) e^{-(1+\varepsilon)x} dx \equiv 0, \; \Re z \geq a. \tag{4.54}$$

Now one can use the inverse theorem for the Laplace transform from Titchmarsh [1] that reduces us to the identity

$$\int_0^x f(x) e^{-(1+\varepsilon)x} dx \equiv 0. \tag{4.55}$$

Hence the desired conclusion of the theorem can be easily obtained from the properties of primaries of summable functions . Thus $f(x) = 0$ for almost every $x \in (0, \infty)$. Theorem 4.12 is proved. •

4.3 The space L^{α} as a normed ring

For our further applications of convolution operator (4.1) to integral equations we need to consider the space L^{α} in view of the theory of the commutative normed rings (see elements of the theory for instance, in Naimark [1]) . Here we illustrate some results from Yakubovich and Luchko [2] for modified convolution (4.1). Our key purpose is to prove the analog of the Wiener theorem about existence of inverse element of the given normed ring. Obviously, the space L^{α} is isometric to $L_1(\mathbf{R}_+)$ and is the Banach space with norm (4.39). Furthermore, owing to Theorem 4.9, one can define an operation of multiplication for elements $f(x)$ and $g(x)$ in the form of convolution (4.1) in the space L^{α}. According to equality (4.4), this operation of the multiplication is commutative in the class L^{α}. Using the Fubini theorem and Theorem 4.12, one can establish its associativity and distributivity, precisely the relations

$$(f*(g*h))(x) = ((f*g)*h)(x), \tag{4.56}$$

$$(f*(g+h))(x) = (f*g)(x) + (f*h)(x) \tag{4.57}$$

for functions $f,\ g,\ h \in L^{\alpha}$. Thus, the space L^{α} forms a commutative Banach ring with the operation of multiplication of elements in the form of convolution (4.1). List some properties of the ring L^{α}.

Theorem 4.13. *The ring L^{α} does not contain the unit with respective to the operation of convolution* (4.1).

Proof. First of all, it is not difficult to show that as a result of convolution of a bounded function with any function from L^{α}, we obtain continuous function for each $x \geq x_0 > 0$. Indeed, let $|g(x)| < C$, where $C > 0$ is a constant, $f(x)$ belong to the ring L^{α} and $h(x) = (f*g)(x)$. Then we have

$$|(f*g)(x)| \leq \frac{C}{2x}\int_0^{\infty}\int_0^{\infty} \exp\left(-\frac{1}{2}\left(\frac{xy}{u}+\frac{xu}{y}+\frac{uy}{x}\right)\right)|f(y)|dydu. \tag{4.58}$$

Calculating the integral by u in (4.58) similarly (4.32) it becomes

$$|(f*g)(x)| \leq C\int_0^{\infty} \frac{yK_1(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}}|f(y)|dy. \tag{4.59}$$

The integral in (4.59) converges absolutely and uniformly for each function $f(x) \in L^{\alpha}$, $\alpha \geq 0$, when $x \geq x_0 > 0$. Indeed, since $f(x) \in L^{\alpha}$, then due to embedding (4.40) $f(x)$ is an element of the space L^0, and by its definition we obtain

$$\int_0^{\infty} \frac{yK_1(\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}}|f(y)|dy$$

$$\leq \int_0^\infty \frac{K_1(\sqrt{x_0^2+y^2})}{K_0(y)} K_0(y)|f(y)|dy \leq C_1||f||_{L^0}, \tag{4.60}$$

where C_1 is a constant, which depends on x_0. Hence, $h(x) = (f*g)(x)$ is a continuous function for $x \geq x_0 > 0$. If the ring L^α contains the unit for the operation of convolution (4.1), then each bounded function from L^α must coincide almost everywhere with some function being continuous for $x \geq x_0 > 0$. Namely, it coincides with the function, which can be obtained as a result of convolution of itself with the unit. But as is evident, that the Lebesgue space L^α contains bounded discontinuous functions, which differ from the functions that are continuous for $x \geq x_0 > 0$ on a set of positive measure. The function, which is equal to 1 on the interval $(a,b), x_0 < a$, and which is equal to zero outside of (a,b) is a simple example. This contradiction shows that the class $L(\mathbf{R}_+; K_\alpha(x))$ does not contain the unit. Theorem 4.13 is proved. •

Let us denote by $V^\alpha \equiv V(\mathbf{R}_+; K_\alpha(x))$ the commutative ring, obtained by means of formal addition of a unit to L^α. Thus, V^α consists of elements $\xi = \lambda e + f(t)$, where e is the unit, λ is an arbitrary complex number, and $f(t)$ is any element from L^α. We introduce the norm in V^α as follows

$$||\xi||_{V^\alpha} = |\lambda| + ||f||_{L^\alpha}. \tag{4.61}$$

Now let us appeal to some preliminary information from the theory of ideals of the rings in Naimark [1].

Definition 4.1. The set I_l of elements of some ring R is called his left ideal, if

1.$I_l \neq R$;

2. From $x, y \in I_l$ it follows that $x + y \in I_l$;

3. From $x \in I_l$, $z \in \mathbf{R}$ it follows that $z \cdot x \in I_l$.

The right ideal is defined analogously.

Definition 4.2. The set I of the elements of the ring R is called the bilateral ideal or ideal in R, if I is the left and the right ideal simultaneously.

Definition 4.3. The bilateral ideal is called the maximal ideal if it is not contained in any other bilateral ideal of the ring R.

The problem arises now to find all maximal ideals of the ring V^α. Directly from Definitions 4.1-4.3 it follows that the ring L^α is a certain maximal ideal in the ring V^α. For each element ξ of the ring V^α let us set by definition

$$\mathcal{F}(s) = \lambda + K_s[f], \tag{4.62}$$

where s is some complex number, $K_s[f]$ is the Kontorovich-Lebedev transform (2.84) of index $s \in C$. With the aid of Theorem 4.10 one can show that the mapping $\xi \to \mathcal{F}(s)$ is a homomorphism of the ring V^α in the field of complex numbers. As is known from Naimark [1] the maximal ideal M_s generated by this mapping contains the set of elements $\xi = \lambda e + f(x)$ of the ring V^α such that $\mathcal{F}(s) = 0$.

The following theorem describes all maximal ideals of the ring V^α.

Theorem 4.14. *There do not exist any maximal ideals in the ring V^α except of L^α and M_s, $s = \mu + i\tau$, where $|\mu| \le \alpha$. Moreover, two ideals M_{s_1} and M_{s_2} coincide if and only if $s_1 = \pm s_2$.*

Proof. Let M is a maximal ideal in V^α, which differs from L^α. Then the mapping

$$F(f) = f(M) \tag{4.63}$$

is a linear functional in V^α. But we can consider it as a linear functional in the space of summable functions with the weight $K_\alpha(x)$, $\alpha \ge 0$. Therefore,

$$f(M) = F(f) = \int_0^\infty f(x)\omega(x)dx, \tag{4.64}$$

where $\frac{\omega(x)}{K_\alpha(x)}$ is essentially bounded function or it belongs to L_∞ space , and $\omega(x) \not\equiv 0$. Since the mapping $f \to f(M)$ is homomorphism, $F(f_1 \cdot f_2) = F(f_1)F(f_2)$, where the operation of multiplication in the ring is the convolution (4.1), that is, we can rewrite this property as follows

$$\int_0^\infty f_1(x)\omega(x)dx \int_0^\infty f_2(y)\omega(y)dy$$
$$= \frac{1}{2}\int_0^\infty \frac{\omega(u)}{u}\int_0^\infty\int_0^\infty \exp\left(-\frac{1}{2}\left(\frac{xy}{u} + \frac{xu}{y} + \frac{uy}{x}\right)\right) f_1(x)f_2(y)dxdydu$$
$$= \int_0^\infty\int_0^\infty f_1(x)f_2(y)\int_0^\infty \exp\left(-\frac{1}{2}\left(\frac{xy}{u} + \frac{xu}{y} + \frac{uy}{x}\right)\right)\frac{\omega(u)}{2u}dudxdy. \tag{4.65}$$

It is clear , that the function $\omega(x)$ must satisfy the following functional equation

$$\omega(x)\omega(y) = \frac{1}{2}\int_0^\infty \exp\left(-\frac{1}{2}\left(\frac{xy}{u} + \frac{xu}{y} + \frac{uy}{x}\right)\right)\frac{\omega(u)}{u}du \tag{4.66}$$

for almost all positive x and y. However, as it follows from the Macdonald formula (1.103), the function $\omega(x) = K_s(x)$, where s is some complex number satisfies equation (4.66). The following lemma describes the solutions of this equation under additional assumption concerning asymptotic behavior of the function $\omega(x)$.

Lemma 4.1. *Let the function $\omega(x)/K_\alpha(x)$ be essentially bounded for $x > 0$ and let $\omega(x) \not\equiv 0$. Then the solution of functional equation (4.66) is the Macdonald function $K_s(x)$, $s = \mu + i\tau$, where $|\mu| \le \alpha$.*

Proof. First show that for any $x_0 > 0, y_0 > 0$ the integral

$$I(x,y) = \int_0^\infty \exp\left(-\frac{1}{2}\left(\frac{xy}{u} + \frac{xu}{y} + \frac{uy}{x}\right)\right)\frac{\omega(u)}{u^\gamma}du, \ \gamma > 1 \tag{4.67}$$

converges uniformly in the region $x \geq x_0 > 0, \ y \geq y_0 > 0$. Indeed,

$$|I(x,y)| = \left|\int_0^\infty \exp\left(-\frac{1}{2}\left(\frac{xy}{u} + \frac{xu}{y} + \frac{uy}{x}\right)\right)\frac{K_\alpha(u)\omega(u)}{u^\gamma K_\alpha(u)}du\right|$$

$$< \int_0^\infty \exp\left(-\frac{1}{2}\left(\frac{xy}{u} + \frac{xu}{y} + \frac{uy}{x}\right)\right)\frac{K_\alpha(u)}{u^\gamma}du\left\|\frac{\omega(u)}{K_\alpha(u)}\right\|_{L_\infty}$$

$$< \left(\int_0^1 \exp\left(-\frac{x_0y_0}{2u}\right)\frac{K_\alpha(u)}{u^\gamma}du + \exp\left(-\frac{x_0y_0}{2}\right)\int_1^\infty \exp(-u)\frac{K_\alpha(u)}{u^\gamma}du\right)$$

$$\times\left\|\frac{\omega(u)}{K_\alpha(u)}\right\|_{L_\infty} < C, \tag{4.68}$$

where $C > 0$ is a constant. It follows, that the integral in the right-hand side of (4.66) can be differentiated with respect to the parameters $x \geq x_0 > 0$ and $y \geq y_0 > 0$. We have

$$\omega'(x)\omega(y) = \frac{1}{2}\int_0^\infty \left(-\frac{u^2+y^2}{2uy} + \frac{uy}{2x^2}\right)\exp\left(-\frac{xy}{2u} - \frac{xu}{2y} - \frac{uy}{2x}\right)\frac{\omega(u)}{u}du$$

$$= \left(\frac{y}{4x^2} - \frac{1}{4y}\right)I_1(x,y) - \frac{y}{4}I_2(x,y), \tag{4.69}$$

where

$$I_1(x,y) = \int_0^\infty \exp\left(-\frac{1}{2}\left(\frac{xy}{u} + \frac{xu}{y} + \frac{uy}{x}\right)\right)\omega(u)du, \tag{4.70}$$

$$I_2(x,y) = \int_0^\infty \exp\left(-\frac{1}{2}\left(\frac{xy}{u} + \frac{xu}{y} + \frac{uy}{x}\right)\right)\frac{\omega(u)}{u^2}du. \tag{4.71}$$

In the similar manner, we also conclude that

$$\omega(x)\omega'(y) = \left(\frac{x}{4y^2} - \frac{1}{4x}\right)I_1(x,y) - \frac{x}{4}I_2(x,y). \tag{4.72}$$

From equalities (4.70)–(4.72) it follows that

$$y\omega'(y)\frac{\omega(x)}{x} = \left(\frac{1}{4y} - \frac{y}{4x^2}\right)I_1(x,y) - \frac{y}{4}I_2(x,y), \tag{4.73}$$

$$x\omega'(x)\frac{\omega(y)}{y} = \left(\frac{1}{4x} - \frac{x}{4y^2}\right)I_1(x,y) - \frac{x}{4}I_2(x,y). \tag{4.74}$$

Differentiating through by y in (4.73) and by x in (4.74), subtract the second equality from the first one. As result we obtain the following differential equation for $\omega(x)$

$$(y\omega'(y))'\frac{\omega(x)}{x} - (x\omega'(x))'\frac{\omega(y)}{y} = \left(\frac{y}{x} - \frac{x}{y}\right)\omega(x)\omega(y), \tag{4.75}$$

which, since the variables x and y are arbitrary, leads to

$$\frac{y\left(y\omega'(y)\right)'}{\omega(y)} - y^2 = \frac{x\left(x\omega'(x)\right)'}{\omega(x)} - x^2 = s^2. \tag{4.76}$$

But both sides in the chain of equalities (4.76) are classical Bessel equations (see for example Erdélyi et al. [1])

$$\omega''(y) + \frac{1}{y}\omega'(y) - \left(1 + \frac{s^2}{y^2}\right)\omega(y) = 0 \tag{4.77}$$

with the solution being the Macdonald function $K_s(y)$, $s = \mu + i\tau$, $|\mu| \leq \alpha$ due to the conditions of the lemma. Thus Lemma 4.1 is proved. •

Hence it is not difficult to obtain the conclusion of Theorem 4.14 . Indeed, since $\omega(x) = K_s(x)$, the maximal ideal M, which is generated by homomorphism (4.62) coincides with M_s due to formula (4.64), where the parameter s, $s = \mu + i\tau$ is defined by the strip $|\mu| \leq \alpha$. By evenness of the Macdonald function $K_s(x)$ with respect to the index it follows that two maximal ideals, which are given by the numbers s_1 and s_2, respectively coincide if and only if $s_1 = \pm s_2$. Thus Theorem 4.14 is proved. •

The mentioned properties of the ring L^α allow us to establish the following analog of the Wiener theorem.

Theorem 4.15. *If the function $\mathcal{F}(s)$, $s = \mu + i\tau$ from relation* (4.62) *does not vanish nowhere in the closed strip $|\mu| \leq \alpha$, including infinity then there is a unique element $q(x)$ from the ring L^α such that*

$$\frac{1}{\mathcal{F}(s)} = \lambda + K_s[q]. \tag{4.78}$$

Proof. As $\mathcal{F}(s)$ does not vanish nowhere in the strip $|\mu| \leq \alpha$, then $f(x)$ does not belong to any maximal ideal, the set of whose is completely exhausted by Theorem 4.14. As it is known from Naimark [1], such an element $f(x)$ has a unique inverse $q(x)$ in the ring V^α, because the mapping $\mathcal{F}(s)$ is a homomorphism. Thus we obtain equality (4.78). This ends the proof of Theorem 4.15. •

Finally, we illustrate the analog of the Titchmarsh theorem (see Titchmarsh [1]) of absence of the divisors of zero for convolution (4.1).

Theorem 4.16. *Let the functions $f(x)$, $g(x)$ be from the ring L^α and $(f*g)(x) \equiv 0$, $x > 0$. Then at least one of the functions $f(x)$ and $g(x)$ is equal to zero almost everywhere on $\mathbf{R}_+$.*

Proof. Indeed, as we noted above equality (4.33) can be extended for the index s, $s = \mu + i\tau$ from the strip $|\mu| \leq \alpha$, precisely

$$K_s[(f*g)] = K_s[f]K_s[g]. \tag{4.79}$$

So we find that the right-hand side of equality (4.79) equals to zero. Since the functions $K_s[f]$, $K_\mu[g]$ are analytic as functions of the complex variable s in the strip $|\mu| < \alpha$, at least one of them is identically zero and Theorem 4.12 leads to the conclusion that at least one of the functions $f(x)$ or $g(x)$ is equal to zero almost everywhere on $\mathbf{R}_+$. Theorem 4.16 is proved. •

4.4 Convolution Hilbert spaces

In this section drawing a parallel with results of Section 2.4 for the Laplace convolution (2.91) we continue to study mapping properties of convolution operator (4.1). We use the weighted spaces L_β^α to define the corresponding convolution Hilbert space by means of completion respective pre-Hilbert space with the inner product as convolution (4.1). The most important results for this purpose are contained in Theorems 4.9-4.10 for subspace L_β^α.

However, let us consider at first the mapping properties of convolution (4.1) at the subspace L_β^α with $0 < \beta < 1$.

Theorem 4.17. *Let $f(x)$, $g(x)$ be from the space L_β^α, $0 < \beta \leq 1$, $\alpha \geq 0$. Then the convolution* (4.1) *exists and belongs to the space L_β^α, moreover*

$$||f*g||_{L_\beta^\alpha} \leq C_\beta ||f||_{L_\beta^\alpha} ||g||_{L_\beta^\alpha}, \tag{4.80}$$

where C_β is a positive constant depending only from β.

Proof. Actually, by the definition of the norm in the space L_β^α we have

$$||f*g||_{L_\beta^\alpha} = \int_0^\infty K_\alpha(\beta x)|(f*g)(x)|dx$$

$$\leq \frac{1}{2}\int_0^\infty \frac{K_\alpha(\beta x)}{x}\int_0^\infty\int_0^\infty \exp\left(-\frac{1}{2}\left(\frac{xy}{u}+\frac{uy}{x}+\frac{xu}{y}\right)\right)|f(u)g(y)|dydudx. \tag{4.81}$$

The inner integral by x corresponds to formula 2.16.9.1 in Prudnikov et al. [1] that gives

$$\frac{1}{2}\int_0^\infty \frac{K_\alpha(\beta x)}{x}\exp\left(-\frac{1}{2}\left(\frac{xy}{u}+\frac{uy}{x}+\frac{xu}{y}\right)\right)dx$$

$$= K_\alpha\left(\frac{1}{2}\left[\sqrt{u^2+y^2+2uy\beta}+\sqrt{u^2+y^2-2uy\beta}\right]\right)$$

$$\times K_\alpha\left(\frac{1}{2}\left[\sqrt{u^2+y^2+2uy\beta}-\sqrt{u^2+y^2-2uy\beta}\right]\right). \tag{4.82}$$

Thus one can estimate the norm of convolution (4.1) as follows

$$\|f*g\|_{L^\alpha_\beta} \le \int_0^\infty \int_0^\infty K_\alpha\left(\frac{1}{2}\left[\sqrt{u^2+y^2+2uy\beta}+\sqrt{u^2+y^2-2uy\beta}\right]\right)$$

$$\times K_\alpha\left(\frac{1}{2}\left[\sqrt{u^2+y^2+2uy\beta}-\sqrt{u^2+y^2-2uy\beta}\right]\right)|f(u)g(y)|dydu. \tag{4.83}$$

Now we need to establish the uniform boundedness of the function of two variables u, v given by formula

$$F(u,v) = \frac{K_\alpha\left(\frac{1}{2}\left[\sqrt{u^2+y^2+2uy\beta}+\sqrt{u^2+y^2-2uy\beta}\right]\right)}{K_\alpha(\beta u)}$$

$$\times\frac{K_\alpha\left(\frac{1}{2}\left[\sqrt{u^2+y^2+2uy\beta}-\sqrt{u^2+y^2-2uy\beta}\right]\right)}{K_\alpha(\beta v)} < C_\beta \tag{4.84}$$

under conditions $0 < \beta \le 1,\ u, v > 0$. In fact, from the asymptotic behavior (1.96)-(1.97) of the Macdonald function it is easily seen that $F(u,v) < C$ for $0 < u, v < \infty$. When $u + v \to \infty$ it is clear that

$$F(u,v) = O\left(\exp\left(\beta(u+v) - \sqrt{u^2+v^2+2uv\beta}\right)\right)$$

$$\le C_\beta \exp\left((\beta - \sqrt{\beta})(u+v)\right) = O(1),\ u+v\to\infty. \tag{4.85}$$

This circumstance allows us to change the order of integration in (4.81) due to Fubini's theorem and to obtain that

$$\|f*g\|_{L^\alpha_\beta} \le \int_0^\infty K_\alpha(\beta u)|f(u)|du$$

$$\times\int_0^\infty K_\alpha(\beta y)|g(y)|dy = \|f\|_{L^\alpha_\beta}\|g\|_{L^\alpha_\beta}. \tag{4.86}$$

This completes the proof of Theorem 4.17. •

One can easily check that for subspaces L^α_β the embedding of type

$$L^\alpha_{\beta_1} \subseteq L^\alpha_{\beta_2},\ \beta_1 \le \beta_2 \tag{4.87}$$

is satisfied. Furthermore, for convolution (4.1) the previous theorem gives the validity of Theorem 4.10, which means that equality (4.33) holds.

Hence to introduce the convolution Hilbert space really more suitable to consider the space L^α_β. Let $\omega(x)$, $x \in \mathbf{R}_+$ be an arbitrary positive function satisfying the conditions

$$\omega(x) \in L_1\left((0,1); \frac{\log x}{x}\right), \quad \omega(x) \in L_1\left((1,\infty); e^{-\beta x}\right), 0 < \beta \leq 1. \tag{4.88}$$

Then one can obtain in view of inequality (1.100) and asymptotic behavior of the Macdonald function that the Kontorovich-Lebedev transform (2.1) of the function $\omega(x)/x$ exists. In fact, we have the estimate

$$\left| K_{i\tau}\left[\frac{\omega(x)}{x}\right]\right| \leq \int_0^\infty |K_{i\tau}(y)| \frac{\omega(y)}{y} dy$$

$$\leq e^{-\delta\tau}\left(\int_0^1 K_0(y\cos\delta)\frac{\omega(y)}{y}dy + \int_1^\infty K_0(y\cos\delta)\frac{\omega(y)}{y}dy\right)$$

$$< e^{-\delta\tau}\left(C_1\int_0^1 \log y \frac{\omega(y)}{y}dy + C_2\int_1^\infty e^{-y\cos\delta}\omega(y)dy\right) < \infty, \tag{4.89}$$

where $\delta \in [0, \pi/2)$ and one can put in (4.88) $\cos\delta = \beta$. In our further considerations we need to impose some additional conditions on the function $\omega(x)$. Precisely speaking, we have to describe some conditions for the positiveness of the Kontorovich-Lebedev transform $K_{i\tau}\left[\frac{\omega(x)}{x}\right]$ for all $\tau \geq 0$. Obviously, from the representation of the Macdonald function $K_{i\tau}(x)$ through integral (1.98) it follows undoubtedly, that this function is real one. Further, by virtue of the composition relation like (4.49) provided that conditions (4.88) are true we have

$$K_{i\tau}\left[\frac{\omega(x)}{x}\right] = \sqrt{\frac{\pi}{2}}\left[F_c\left[L\frac{\omega(x)}{x}\right](\cosh u)\right](\tau), \tag{4.90}$$

where we mean $[F_c f](u)$ as the cosine-Fourier transform (1.197). Recall now to one useful theorem from Titchmarsh [1] (see Theorem 124) to provide some sufficient conditions for the positiveness of the composition (4.90). Indeed, by assumptions (4.88) the Laplace transform (1.215) of the function $\omega(x)/x$ depends upon variable $\cosh u$, $u \geq 0$ and it is a bounded function on $\mathbf{R}_+$. In addition, it steadily decreases to zero as u diverges to infinity. In fact, for $u_1 > u_2$ we obtain

$$L\left\{\frac{\omega(x)}{x}; \cosh u_1\right\} = \int_0^\infty \frac{\omega(y)}{y} e^{-y\cosh u_1} dy$$

$$< \int_0^\infty \frac{\omega(y)}{y} e^{-y\cosh u_2} dy < C_1\int_0^1 \log y \frac{\omega(y)}{y}dy + C_2\int_1^\infty e^{-y\beta}\omega(y)dy < \infty, \tag{4.91}$$

and this Laplace transform tends to 0 as $u \to +\infty$ by the Lebesgue theorem. Moreover, we can differentiate this integral by u under the integral sign , namely

$$\frac{d}{du}L\left\{\frac{\omega(x)}{x}; \cosh u\right\} = -\sinh u\int_0^\infty \omega(y)e^{-y\cosh u}dy. \tag{4.92}$$

Thus we arrived to the following result.

Theorem 4.18. *Let conditions* (4.88) *be true for the function* $\omega(x)$ *and the integral in the right-hand side of equality* (4.92) *be positive, non-increasing, and tend to a limit at infinity. Then the Kontorovich-Lebedev transform* (4.90) *is positive function for all* $\tau \geq 0$.

Proof. The proof of this theorem immediately follows from the mentioned Theorem 124 in Titchmarsh [1]. Namely, under the above conditions the Laplace transform $[L\frac{\omega(x)}{x}](\cosh u)$ is a bounded function, which decreases steadily to zero at infinity both with integral (4.92). Note, that we assumed its positiveness and respective monotonicity under conditions of the theorem. Thus we conclude that the Laplace transform $[L\frac{\omega(x)}{x}](\cosh u)$ is convex downwards function of variable u. Therefore, composition (4.90) with the cosine Fourier transform is positive for $\tau \geq 0$. Theorem 4.18 is proved. •

One can consider now several concrete examples of the function $\omega(x)$ and the corresponding Kontorovich-Lebedev transforms (2.1). Evidently, the function $\omega(x) \equiv x$ satisfies conditions (4.88). In this case invoking with formula 2.16.2.1 in Prudnikov et al. [2], we obtain the following expression for the Kontorovich-Lebedev transform

$$\int_0^\infty K_{i\tau}(y)dy = \frac{\pi}{2\cosh(\pi\tau/2)}. \tag{4.93}$$

Further, letting for instance, $\omega(x) \equiv x^\gamma$, $\gamma > 0$ write integral 2.16.2.2 in Prudnikov et al. [2] as

$$\int_0^\infty y^{\gamma-1} K_{i\tau}(y)dy = 2^{\gamma-2} \left|\Gamma\left(\frac{\gamma + i\tau}{2}\right)\right|^2. \tag{4.94}$$

The next example is given by the function $\omega(x) \equiv e^{-x}x^\gamma$, $\gamma > 0$. We appeal to integral 2.16.6.4 from the same item and obtain

$$\int_0^\infty y^{\gamma-1} e^{-y} K_{i\tau}(y)dy = 2^{-\gamma}\sqrt{\pi}\frac{|\Gamma(\gamma + i\tau)|^2}{\Gamma(\gamma+1/2)}. \tag{4.95}$$

The last example here deals with the function $\omega(x) \equiv xe^{-\gamma x}$, $0 \leq \gamma < 1$. Making use formula 2.16.6.1 in Prudnikov et al. [2] we have

$$\int_0^\infty e^{-\gamma y} K_{i\tau}(y)dy = \frac{\pi \sinh(\tau \cos^{-1}\gamma)}{\sinh(\pi\tau)\sqrt{1-\gamma^2}}. \tag{4.96}$$

Note that a series of such examples can be written by using the integrals in the mentioned volume of Prudnikov et al. [2] as well as of the convolution and the index types.

Thus we consider below the function

$$q(\tau) = K_{i\tau}\left[\frac{\omega(x)}{x}\right], \ \tau \geq 0, \tag{4.97}$$

as the weighted function for the respective Lebesgue spaces. Take in general two complex-valued functions $f(x)$ and $g(x)$ from the space $L^{\alpha}_{\cos\delta} \subset L^{\alpha}$, $\pi/3 < \delta < \pi/2$, $\alpha \geq 0$. Then according to Theorem 4.17 the convolution $(f * \bar{g})(x)$ exists and belongs to space $L^{\alpha}_{\cos\delta}$. Moreover, factorization equality (4.33) holds and due to Theorem 4.11 Parseval's relation (4.34) holds. Hence multiply through in (4.34) by $\omega(x)$ and integrate by $\mathbf{R}_+$ we obtain the following equality

$$\int_0^{\infty} (f * \bar{g})(x)\omega(x)dx$$

$$= \frac{2}{\pi^2}\int_0^{\infty} \frac{\omega(x)dx}{x}\int_0^{\infty} \tau\sinh(\pi\tau)K_{i\tau}(x)K_{i\tau}[f]K_{i\tau}[\bar{g}]d\tau. \tag{4.98}$$

Making use estimate (4.46) and conditions (4.88) for the function $\omega(x)$ it follows that

$$\int_0^{\infty} |(f * \bar{g})(x)|\omega(x)dx$$

$$\leq \frac{2}{\pi^2}\int_0^{\infty} K_0(x\cos\delta)\frac{\omega(x)}{x}dx\int_0^{\infty}\tau\sinh(\pi\tau)e^{-3\tau\delta}d\tau$$

$$\times \int_0^{\infty} K_0(u\cos\delta)|f(u)|du\int_0^{\infty} K_0(y\cos\delta)|g(y)|dy \tag{4.99}$$

and all integrals are convergent under conditions above. Thus finally apply the Fubini theorem that enables us to change the order of integration at the right-hand side of equality (4.98). So we come to the equality

$$\int_0^{\infty} (f * \bar{g})(x)\omega(x)dx$$

$$= \frac{2}{\pi^2}\int_0^{\infty}\tau\sinh(\pi\tau)q(\tau)K_{i\tau}[f]K_{i\tau}[\bar{g}]d\tau, \tag{4.100}$$

where the weighted function $q(\tau)$ is defined by formula (4.97). Denote the left-hand side of (4.100) by

$$\int_0^{\infty} (f * \bar{g})(x)\omega(x)dx = \langle f, g\rangle. \tag{4.101}$$

From equality (4.101) and Theorem 4.12 observe that $\langle f, g\rangle$ possesses by all properties of the inner product. With this inner product the set of functions $L^{\alpha}_{\cos\delta}$ becomes the pre-Hilbert space. Its completion we shall call as the convolution Hilbert space and shall denote it by S_q. So for any elements $f \in S_q$, $g \in S_q$ the inner product $\langle f, g\rangle$ is defined as well as the norm $||f||_S = \sqrt{\langle f, f\rangle}$. If $f \in L^{\alpha}_{\cos\delta}$, $g \in L^{\alpha}_{\cos\delta}$, then we have

$$\langle f, g\rangle = \int_0^{\infty} (f * \bar{g})(x)\omega(x)dx$$

$$= \frac{1}{2}\int_0^{\infty}\frac{\omega(x)}{x}dx\int_0^{\infty}\int_0^{\infty}\exp\left(-\frac{1}{2}\left[\frac{xu}{y}+\frac{xy}{u}+\frac{yu}{x}\right]\right)f(u)\overline{g(y)}dudy$$

$$= \int_0^{\infty}\int_0^{\infty} S_{\omega}(u, y)f(u)\overline{g(y)}dudy, \tag{4.102}$$

where

$$S_\omega(u,y) = \frac{1}{2}\int_0^\infty \frac{\omega(x)}{x} \exp\left(-\frac{1}{2}\left[\frac{xu}{y} + \frac{xy}{u} + \frac{yu}{x}\right]\right) dx. \tag{4.103}$$

The change of the integration order in (4.102) can be performed by the above estimate and the Fubini theorem. Thus if $f(x)$ satisfies the condition

$$\int_0^\infty \int_0^\infty S_\omega(u,y)|f(u)f(y)|dudy < \infty, \tag{4.104}$$

then $||f||_S < \infty$ and $f \in S_q \supset L^\alpha_{\cos\delta}$. On the other hand, if $f(x)$ and $g(x)$ satisfy condition (4.104), then the Cauchy-Schwarz-Bunyakovskii inequality implies

$$|\langle f, g\rangle| \leq ||f||_S ||g||_S, \tag{4.105}$$

and it gives us that the integral

$$\int_0^\infty \int_0^\infty S_\omega(u,y)|f(u)g(y)|dudy \tag{4.106}$$

is convergent and equality (4.102) is valid.

Let us denote through $H_q \equiv L_2(\mathbf{R}_+; \frac{2}{\pi^2}\tau\sinh(\pi\tau)q(\tau))$ the weighted Hilbert space of functions $h(\tau)$ with the norm

$$||h||_{H_q} = \frac{\sqrt{2}}{\pi}\left(\int_0^\infty \tau\sinh(\pi\tau)q(\tau)|h(\tau)|^2 d\tau\right)^{1/2}, \tag{4.107}$$

where $q(\tau)$ is the weighted function (4.97). As it follows from (4.100) the operator of the Kontorovich-Lebedev transform (2.1) maps the space $L^\alpha_{\cos\delta}$ into H_q and moreover,

$$\begin{aligned} ||K_{i\tau}[f]||^2_{H_q} &= \frac{2}{\pi^2}\int_0^\infty \tau\sinh(\pi\tau)q(\tau)|K_{i\tau}[f]|^2 d\tau \\ &= \int_0^\infty (f * \bar{f})(x)\omega(x)dx = ||f||^2_S. \end{aligned} \tag{4.108}$$

According to the Banach Theorem 1.5 extend by continuity the Kontorovich-Lebedev operator for all $f \in S_q$. So the Kontorovich-Lebedev transform is defined for all $f \in S_q$, its range $KL(S_q)$ belongs to H_q and for any $f \in S_q$ we obtain

$$||f||_S = ||K_{i\tau}[f]||_{H_q},\ K_{i\tau}[f] = 0, \text{ iff } f = 0. \tag{4.109}$$

Consequently, one can conclude that there exists the inverse bounded operator $K^{-1}_{ix}[h]$. By virtue of the definition of norm (4.109) and equality (4.102) we naturally write the inner product of two elements φ, ψ at the space H_q by formula

$$(\varphi, \psi) = \frac{2}{\pi^2}\int_0^\infty \tau\sinh(\pi\tau)q(\tau)\varphi(\tau)\overline{\psi(\tau)}d\tau. \tag{4.110}$$

Returning to the considered examples of function $\omega(x)$ we immediately obtain the respective examples of the Hilbert spaces S_q and H_q with relation (4.100) and condition (4.104) for each case. For instance, putting $\omega(x) \equiv x$ we use formula (4.93) and equality (4.100) becomes

$$\int_0^\infty (f * \bar{g})(x)xdx$$

$$= \frac{2}{\pi} \int_0^\infty \tau \sinh(\pi\tau/2) K_{i\tau}[f] K_{i\tau}[\bar{g}] d\tau. \tag{4.111}$$

To deduce the corresponding condition (4.104) we need to calculate integral (4.103) by means of formula (4.27). Hence we have

$$\int_0^\infty \int_0^\infty \frac{uy}{\sqrt{u^2+y^2}} K_1(\sqrt{u^2+y^2}) |f(u)f(y)| du dy < \infty. \tag{4.112}$$

Similarly we can treat other examples (4.94)-(4.96).

As we noted above if $f(x),\ g(x) \in L^\alpha_{\cos\delta}$ then $(f*g)(x) \in L^\alpha_{\cos\delta}$ and equality (4.33) is true. Hence the following representation holds

$$(f*g)(x) = K^{-1}_{ix} \left[K_{ix}[f] K_{ix}[g] \right]. \tag{4.113}$$

Therefore, if for two elements of convolution Hilbert space S_q f, g the function $K_{i\tau}[f]K_{i\tau}[g] = \varphi(\tau)\psi(\tau) \in KL(S_q)$, then the element $K^{-1}_{ix}[\varphi\psi]$ it is naturally to call the generalized convolution of the elements f, g and to denote by $(f*g)$. Let us prove that for any $f \in S_q$ and $g \in L^\alpha_{\cos\delta}$ convolution (4.1) exists and the inequality of type

$$||f*g||_S \le \sup_{\tau>0} |\psi(\tau)|\ ||f||_S, \tag{4.114}$$

holds , where

$$\psi(\tau) \equiv K_{i\tau}[g] = \int_0^\infty K_{i\tau}(y) g(y) dy. \tag{4.115}$$

Indeed, since the function $g(x) \in L^\alpha_{\cos\delta}$ then owing to the estimate

$$|\psi(\tau)| \le e^{-\delta\tau} \int_0^\infty K_0(y\cos\delta)|g(y)| dy \tag{4.116}$$

we obtain that $\sup_{\tau>0}|\psi(\tau)| = M < \infty$ and consequently, $\varphi(\tau)\psi(\tau) \in H_q$, $\varphi(\tau) \equiv K_{i\tau}[f]$. Let us prove now that $\varphi(\tau)\psi(\tau) \in KL(S_q)$. There exists some sequence $f_n(x) \in L^\alpha_{\cos\delta}$ such that $||f - f_n||_S$ tends to zero as n tends to infinity. Hence according to Theorem 4.17 $h_n(x) = (f_n * g)(x) \in L^\alpha_{\cos\delta}$ and denoting by $\varphi_n(\tau) = K_{i\tau}[f_n]$ we have (see (4.107))

$$||h_n - h_m|| = ||(f_n - f_m)*g||_S = ||K_{i\tau}[f_n - f_m]K_{i\tau}[g]||_{H_q}$$

$$= ||(\varphi_n - \varphi_m)\psi||_{H_q} \le M||\varphi_n - \varphi_m||_{H_q} = M||f_n - f_m||_S. \tag{4.117}$$

It is clear now, that the sequence h_n is convergent at the Hilbert space S_q. Let the corresponding limit be h. Then

$$K_{i\tau}[h] = K_{i\tau}[f]K_{i\tau}[g] = \varphi(\tau)\psi(\tau). \tag{4.118}$$

Thus the product $\varphi(\tau)\psi(\tau)$ belongs to $KL(S_q)$.

We turn now to establish inversion the formula for the Kontorovich-Lebedev transform in the convolution Hilbert space.

Theorem 4.19. *Let the weighted function $\omega(x)$ be satisfied conditions* (4.88) *with $\beta = \cos\delta$, $\pi/3 < \delta < \pi/2$. Then for functions f from the convolution Hilbert space S_q for all $x > 0$ the following inversion formula of the Kontorovich-Lebedev transform is fulfilled*

$$\left(f * \frac{\omega(x)}{x}\right)(x) = \frac{2}{\pi^2 x}\frac{d}{dx}\int_0^\infty \tau\sinh(\pi\tau)q(\tau)$$

$$\times\Re_{i\tau}\left[\frac{x^{1-i\tau}2^{i\tau}\Gamma(i\tau)}{1-i\tau}{}_1F_2\left(\frac{1-i\tau}{2};\ 1-i\tau,\ \frac{3-i\tau}{2};\ \frac{x^2}{4}\right)\right]K_{i\tau}[f]d\tau, \tag{4.119}$$

where the symbol $\Re_{i\tau}$ is described, for instance, in (1.163). *In addition, if $f(x) \in L^{\alpha}_{\cos\delta} \subset S_q$, then formula* (4.119) *takes like classical form*

$$(f * \frac{\omega(x)}{x})(x) = \frac{2}{\pi^2}\int_0^\infty \tau\sinh(\pi\tau)q(\tau)\frac{K_{i\tau}(x)}{x}K_{i\tau}[f]d\tau. \tag{4.120}$$

Proof. In order to prove formula (4.119) we start from relation (4.100). Letting there $g(y) = 1,\ 0 < y \le x;\ g(y) = 0,\ y > x$ we transform the right-hand side of equality (4.100), calculating the respective integral $K_{i\tau}[g]$ by formula (2.71) (repeat it again for our convenience), namely

$$\int_0^x K_{i\tau}(y)dy = \Re_{i\tau}\left[\frac{x^{1-i\tau}2^{i\tau}\Gamma(i\tau)}{1-i\tau}{}_1F_2\left(\frac{1-i\tau}{2};\ 1-i\tau,\ \frac{3-i\tau}{2};\ \frac{x^2}{4}\right)\right]. \tag{4.121}$$

Clearly, that under conditions of the theorem and owing to inequality (4.105) the left-hand side of equality (4.100) is an absolutely convergent integral. So invoking with notation (4.103) formula (4.100) becomes as

$$\int_0^x\int_0^\infty S_\omega(u,y)f(u)dudy$$

$$= \frac{2}{\pi^2}\int_0^\infty \tau\sinh(\pi\tau)q(\tau)$$

$$\times\Re_{i\tau}\left[\frac{x^{1-i\tau}2^{i\tau}\Gamma(i\tau)}{1-i\tau}{}_1F_2\left(\frac{1-i\tau}{2};\ 1-i\tau,\ \frac{3-i\tau}{2};\ \frac{x^2}{4}\right)\right]K_{i\tau}[f]d\tau. \tag{4.122}$$

Moreover, this enables us to perform differentiation through by x in (4.122) to obtain

$$\int_0^\infty S_\omega(u,x)f(u)du$$

$$= \frac{2}{\pi^2}\frac{d}{dx}\int_0^\infty \tau\sinh(\pi\tau)q(\tau)$$

$$\times\Re_{i\tau}\left[\frac{x^{1-i\tau}2^{i\tau}\Gamma(i\tau)}{1-i\tau}{}_1F_2\left(\frac{1-i\tau}{2};\ 1-i\tau,\ \frac{3-i\tau}{2};\ \frac{x^2}{4}\right)\right]K_{i\tau}[f]d\tau. \tag{4.123}$$

However, the left-hand side of (4.123) equals $x(f * \frac{\omega(x)}{x})(x)$ which leads to (4.119). Hence formula (4.120) can be easily deduced performing the differentiation under the integral by τ in the right-hand side of (4.123). We motivate it meaning that $f(x) \in L^{\alpha}_{\cos\delta}$ and invoke with inequality (4.46) where instead of g we set the function $\omega(x)/x \in L^{\alpha}_{\cos\delta}$ under condition of the present theorem. This completes the proof of Theorem 4.19. •

Putting in the formula (4.119) $\omega(x) \equiv x$ we attract our attention to the corresponding inversion formula for the space (4.112), namely

$$(f * 1)(x) = \frac{2}{\pi x}\frac{d}{dx}\int_0^\infty \tau \sinh(\pi\tau/2)$$

$$\times \Re_{i\tau}\left[\frac{x^{1-i\tau}2^{i\tau}\Gamma(i\tau)}{1-i\tau}{}_1F_2\left(\frac{1-i\tau}{2};\ 1-i\tau,\ \frac{3-i\tau}{2};\ \frac{x^2}{4}\right)\right] K_{i\tau}[f]d\tau. \qquad (4.124)$$

Note that if the range of the Kontorovich-Lebedev transform $KL(S_q)$ coincides with the space H_q, then for existence of the convolution $(f * g)(x)$ of elements $f, g \in S_q$ it is necessary and sufficiently that the product $K_{i\tau}[f]K_{i\tau}[\bar{g}]$ belongs to the space H_q. However, it is true that $KL(S_q) = H_q$.

Theorem 4.20. *The range of the Kontorovich-Lebedev transform $KL(S_q)$ coincides with the weighted Hilbert space H_q.*

Proof. In fact, in this case there no exists in H_q any element except zero that is orthogonal to $KL(S_q)$. Precisely, let $(\varphi_0,\ K_{i\tau}[g]) = 0$ for arbitrary $g \in S_q$, where the inner product means here by formula (4. 110). In particular, take the function g as in Theorem 4.19 $g(y) = 1,\ 0 < y \le x;\ g(y) = 0,\ y > x$. However, the equality

$$\left(\varphi_0,\ \int_0^x K_{i\tau}(y)dy\right) = \frac{2}{\pi^2}\int_0^\infty \tau\sinh(\pi\tau)q(\tau)\varphi_0(\tau)\int_0^x K_{i\tau}(y)dyd\tau = 0 \qquad (4.125)$$

after differentiation by x yields

$$\int_0^\infty \tau\sinh(\pi\tau)q(\tau)\varphi_0(\tau)K_{i\tau}(x)d\tau = 0 \qquad (4.126)$$

for all $x > 0$. The last operation is possible due to absolute and uniform convergence of integral (4.126) in view of the estimate

$$\int_0^\infty \tau\sinh(\pi\tau)q(\tau)|\varphi_0(\tau)K_{i\tau}(x)|d\tau$$

$$\le \frac{\pi K_0(x\cos\delta)}{\sqrt{2}}||\varphi_0||_{H_q}\left(\int_0^\infty \tau\sinh(\pi\tau)q(\tau)e^{-2\delta\tau}d\tau\right)^{1/2}. \qquad (4.127)$$

So estimate (4.127) shows that the left hand-side of (4.126) is a function from $L_1(\mathbf{R}_+)$ and according to the theory of Fourier integrals in Titchmarsh [1] one can apply through in equality (4.126) the cosine Fourier transform (1.197). Changing the order of integration by the Fubini theorem and calculating the inner integral by formula 2.16.14.1 in Prudnikov et al. [2] we obtain new equality as

$$\int_0^\infty \tau\sinh(\pi\tau/2)q(\tau)\varphi_0(\tau)\cos\left(\tau\log(x+\sqrt{x^2+1})\right)d\tau \equiv 0. \qquad (4.128)$$

Hence, observing that owing to the above estimates the integrand in (4.128) belongs to the space $L_1(\mathbf{R}_+)$ by τ (one can verify it by using the Hölder inequality like in (4.127)), appeal to the familiar property concerning the uniqueness of the cosine Fourier transform of summable functions from $L_1(\mathbf{R}_+)$ (see Titchmarsh [1]). It gives us that $\varphi_0(\tau) = 0$ almost everywhere. Theorem 4.20 is proved. •

4.5 On the Kontorovich-Lebedev convolution integral equations

This section deals with the decision problem of integral equations of the first and the second kind which involve the kernel (4.3) and contain the inner integral of the Kontorovich-Lebedev convolution (4.1). Such equations were first mentioned in Lebedev [6] and were exhibited in detail recently in Yakubovich [4]-[5], Yakubovich and Luchko [2]. Comparing with usual convolution equations of the Fourier, the Mellin or the Laplace type (see Titchmarsh [1], Srivastava and Buschman [1], Prudnikov et al. [5]) for operator (4.2) it is not so easily to recognize its convolution properties. Nevertheless, this class of integral equations is also worth mentioning in connection with some applications to problems of mathematical physics (see Lebedev [6]-[7], [9]). First as is shown in Yakubovich [4] these equations one can solve using the algebra of the introduced convolution (4.1). Here we give some examples of the Kontorovich-Lebedev type convolution integral equations and their solutions in slightly different form than in Yakubovich and Luchko [2] in view of the considered convolution operator (4.1) and its new properties. Operational method of solution of such equations has been developed recently in Yakubovich and Luchko [2]-[3].

As is known the most familiar form of integral equation is

$$f(x) = h(x) + \lambda \int_0^\infty \mathcal{K}(x,u) f(u) dy, \ \ x > 0, \tag{4.129}$$

where λ is some complex parameter, $h(x)$ and $\mathcal{K}(x,u)$ are given functions, and $f(x)$ is to be determined. We shall call such equation as usually the integral equation of the second kind. It can be solved by means of the Kontorovich-Lebedev integrals in certain special cases meaning such integral equations in which the integral operator (4.129) is the convolution operator (4.2) with kernel (4.3).

Let us consider some examples of explicit kernels (4.3) choosing different functions $g(x)$ and calculating the respective integrals. Let $g(x) = \exp(-x\cos\mu)x^{\gamma-1}$, $0 \le \mu < \pi$, $\Re\gamma > 0$. Then due to formula 2.3.16.1 in Prudnikov et al. [1], the function $\mathcal{K}(x,u)$ equals

$$\mathcal{K}(x,u) = \frac{x^{\gamma-1}u^{\gamma}}{(x^2+u^2+2xu\cos\mu)^{\gamma/2}} K_\gamma(\sqrt{x^2+u^2+2xu\cos\mu}) \tag{4.130}$$

and equation (4.129) takes the form

$$f(x) = h(x) + \lambda \int_0^\infty \frac{x^{\gamma-1} u^\gamma K_\gamma(\sqrt{x^2+u^2+2xu\cos\mu})}{(x^2+u^2+2xu\cos\mu)^{\gamma/2}} f(u)du. \tag{4.131}$$

Setting in (4.131) $\gamma = 1/2$ and invoking with the fact that the Macdonald function $K_{1/2}(z)$ reduces to

$$K_{1/2}(z) = e^{-z}\sqrt{\frac{\pi}{2z}}, \tag{4.132}$$

rewrite equation (4.132) for this case as

$$f(x) = h(x) + \lambda\sqrt{\frac{\pi}{2x}} \int_0^\infty \frac{\exp\left(-\sqrt{x^2+u^2+2xu\cos\mu}\right)}{(x^2+u^2+2xu\cos\mu)^{1/2}} u^{1/2} f(u)du. \tag{4.133}$$

Conversely, the simplest case of equation (4.133) was considered first in Lebedev [6], when $\mu = 0$, namely

$$f(x) = h(x) + \lambda\sqrt{\frac{\pi}{2x}} \int_0^\infty \frac{\exp(-x-u)}{x+u} u^{1/2} f(u)du. \tag{4.134}$$

Let $g(x) = (x+a)^{-1}$, where $a > 0$ is a parameter. Then integral (4.3) can be evaluated by formula 2.3.16.4 Prudnikov et al. [1] as follows

$$\mathcal{K}(x,u) = \frac{\sqrt{\pi/a}}{2x} \exp\left(a\frac{x^2+u^2}{2xu} + \frac{xu}{2a}\right) \operatorname{erfc}\left(\sqrt{\frac{xu}{2a}} + a\frac{\sqrt{x^2+u^2}}{2xu}\right), \tag{4.135}$$

where

$$\operatorname{erfc}(x) = \frac{2}{\pi}\int_x^\infty e^{-t^2} dt \tag{4.136}$$

is the error function (see Erdélyi et al. [1]).

Return to the general convolution operator (4.2). We already noted by Corollary 4.1 its mapping properties and behavior of the kernel $\mathcal{K}(x,u)$ at the neighborhood of the point $(0,0)$. The examples of the kernel $\mathcal{K}(x,u)$ considered above confirm that it contains immovable singularity at the origin $(0,0)$.

We begin from the following homogeneous equation

$$f(x) = \lambda(\tau)\int_0^\infty \mathcal{K}(x,u) f(u)du, \ x > 0, \tag{4.137}$$

where $\lambda(\tau)$ is a continuous function on $\mathbf{R}_+$ of variable τ.

Theorem 4.21. *Let $g(x) \in L^0 \equiv L(\mathbf{R}_+; K_0(x))$. If $1/\lambda(\tau) = K_{i\tau}[g]$, where $K_{i\tau}[g]$ is the Kontorovich-Lebedev transform* (2.1)*, then the function $K_{i\tau}(x)/x$ satisfies equation* (4.137).

Proof. Substituting $K_{i\tau}(x)/x$ in the right-hand side of equality (4.137) and taking into account inequality (1.147), we obtain the estimate

$$\left|\lambda(\tau)\int_0^\infty \mathcal{K}(x,u)K_{i\tau}(u)\frac{du}{u}\right| < |\lambda(\tau)|\int_0^\infty |\mathcal{K}(x,u)|\frac{K_0(u)}{u}du. \tag{4.138}$$

Due to the Macdonald formula (1.103), continue inequality (4.138) as follows

$$\int_0^\infty |\mathcal{K}(x,u)|K_0(u)\frac{du}{u} < \frac{K_0(x)}{x}\int_0^\infty K_0(y)|g(y)|dy = \frac{K_0(x)}{x}||g||_{L^0}. \tag{4.139}$$

Hence we perform to change the order of integration by Fubini's theorem and obtain identity (4.137) with $f(x) = K_{i\tau}(x)/x$. Theorem 4.21 is proved. •

The partial solution of equation (4.129) is given by the following

Theorem 4.22. *Let $h(x),\ g(x)\ \in L_p(\mathbf{R}_+)$, $p \geq 1$ and $\sup\limits_{\tau\geq 0}|K_{i\tau}[g]| < \frac{1}{|\lambda|}$. Then the function*

$$f(x) = \frac{2}{\pi^2}\text{l.i.m.}_{\varepsilon\to 0+}\int_0^\infty \frac{\tau\sinh((\pi-\varepsilon)\tau)K_{i\tau}[h]}{1-\lambda K_{i\tau}[g]}\frac{K_{i\tau}(x)}{x}d\tau \tag{4.140}$$

is a partial L_p-solution of equation (4.129) *and the limit is meant by the norm of $L_p(\mathbf{R}_+)$.*

Proof. First one can show that the space $L_p(\mathbf{R}_+)$, $p \geq 1$ is a subspace of the space $L^0_{\cos\delta}$, $\delta \in (0,\pi/2)$. Indeed, if $g \in L_p(\mathbf{R}_+)$, then by using the Hölder inequality (1.8) we have

$$\int_0^\infty K_0(y\cos\delta)|g(y)|dy \leq \left(\int_0^\infty K_0^q(y\cos\delta)dy\right)^{1/q}||g||_{L_p} < \infty \tag{4.141}$$

which gives the desired result. Conversely, according to Theorem 2.2 for the function $h \in L_p(\mathbf{R}_+)$ the limit relation is true

$$h(x) = \frac{2}{\pi^2}\text{l.i.m.}_{\varepsilon\to 0+}\int_0^\infty \tau\sinh((\pi-\varepsilon)\tau)\frac{K_{i\tau}(x)}{x}K_{i\tau}[h]d\tau. \tag{4.142}$$

In addition, due to the fact $h(x) \in L^0_{\cos\delta}$ the following estimate holds

$$|K_{i\tau}[h]| \leq e^{-\delta\tau}||h||_{L^0_{\cos\delta}},\ \delta\in(0,\pi/2). \tag{4.143}$$

Hence denoting the right-hand side of (4.142) as $(I_\varepsilon f)(x)$ under condition of this theorem for $K_{i\tau}[g]$ we obtain that for each $\varepsilon > 0$

$$|(I_\varepsilon f)(x)| \leq C\frac{K_0(x\cos\delta)}{x}||h||_{L^0_{\cos\delta}}\int_0^\infty \tau\sinh((\pi-\varepsilon)\tau)e^{-2\delta\tau}d\tau, \tag{4.144}$$

where C is an absolute positive constant and δ is chosen from the interval $((\pi - \varepsilon)/2,\ \pi/2)$. Therefore there exists the convolution (4.1) $(g*(I_\varepsilon f))(x)$ provided by the estimate

$$|(g*(I_\varepsilon f))(x)| < \frac{C}{x}||h||_{L^0_{\cos\delta}}\int_0^\infty |g(y)|dy$$

$$\times\int_0^\infty \exp\left(-\frac{1}{2}\left(\frac{xu}{y}+\frac{xy}{u}+\frac{yu}{x}\right)\right)\frac{K_0(u\cos\delta)}{u}du$$

$$< C\frac{K_0(x\cos\delta)}{x}||h||_{L^0_{\cos\delta}}\int_0^\infty K_0(y\cos\delta)|g(y)|dy$$

$$\leq C_\delta\frac{K_0(x\cos\delta)}{x}||h||_{L^0_{\cos\delta}}||g||_p, \tag{4.145}$$

where C, C_δ are positive constants. To establish this estimate we used the inequality (4.84) as well as the Hölder inequality (1.8). Hence it is clear that

$$\frac{K_{i\tau}[h]}{1-\lambda K_{i\tau}[g]} \in KL(L_p), \tag{4.146}$$

taking into account the conclusion of Theorem 2.3, namely, that conditions (2.26)-(2.27) hold. Moreover, invoking with the Macdonald formula (1.103), Theorem 2.2 and the Lebesgue Theorem 1.2 we have almost everywhere the limit equality

$$\lim_{\varepsilon\to 0+}\lambda(g*(I_\varepsilon f))(x) = (f*g)(x)$$

$$= \lim_{\varepsilon\to 0+}\frac{2\lambda}{\pi^2 x}\int_0^\infty \frac{\tau\sinh((\pi-\varepsilon)\tau)K_{i\tau}[h]}{1-\lambda K_{i\tau}[g]}K_{i\tau}[g]K_{i\tau}(x)d\tau$$

$$= -\lim_{\varepsilon\to 0+}\frac{2}{\pi^2 x}\int_0^\infty \tau\sinh((\pi-\varepsilon)\tau)K_{i\tau}(x)K_{i\tau}[h]d\tau$$

$$+\lim_{\varepsilon\to 0+}\frac{2}{\pi^2 x}\int_0^\infty \frac{\tau\sinh((\pi-\varepsilon)\tau)K_{i\tau}[h]}{1-\lambda K_{i\tau}[g]}K_{i\tau}(x)d\tau = -h(x)+f(x), \tag{4.147}$$

where the last equality it is easy to see from Theorem 2.2. This ends the proof of Theorem 4.22. •

Now consider an equation similar to (4.129) with the kernel (4.3) and $\lambda = -1$

$$h(x) = f(x) + \int_0^\infty \mathcal{K}(x,u)f(u)du \tag{4.148}$$

with respect to the function $f(x)$ from the space $L(\mathbf{R}_+; K_\alpha(x))$, where the given functions $h(x)$ and $g(x)$ in (4.3) belong to the normed ring $L(\mathbf{R}_+; K_\alpha(x))$. Applying through by the Kontorovich-Lebedev transform (2.84) with index $s = \mu + i\tau$ from the strip $|\mu| \leq \alpha$ in equation (4.148) from factorization equality (4.79) for the convolution $(f*g)(x)$ deduce the following algebraic equation

$$K_s[h] \;=\; K_s[f](1+K_s[g]),\; |\mu|\leq\alpha. \tag{4.149}$$

If the condition

$$1+K_s[g]\neq 0,\; |\mu|\leq\alpha, \tag{4.150}$$

holds, then by the analog of the Wiener Theorem 4.15, there is a unique function $q(x)\in L^\alpha$ such that

$$\frac{1}{1+K_s[g]} \;=\; 1+K_s[q]. \tag{4.151}$$

Hence we obtain the equality

$$K_s[f] = (1 + K_s[g])^{-1} K_s[h]$$

$$= (1 + K_s[q]) K_s[h], \ |\mu| \leq \alpha, \tag{4.152}$$

which is equivalent to

$$f(x) = h(x) + \int_0^\infty \mathcal{K}_q(x,u) h(u) du, \ x > 0, \tag{4.153}$$

where $\mathcal{K}_q(x,u)$ is a new kernel like (4.3) with the function $q(x)$. It is easily seen, that conversely, the function $f(x)$ from formula (4.153) gives the solution of the equation (4.148) for any function $h(x)$ from the ring L^α only under condition (4.150).

Thus we proved the following

Theorem 4.23. *Let functions $g(x)$ and $h(x)$ belong to the class $L(\mathbf{R}_+; K_\alpha(x))$. Then equation* (4.148) *is solvable in the class L^α if and only if condition* (5.150) *holds. Moreover, its unique solution is represented by formula* (4.153).

Corollary 4.2. *Equation* (4.131) *with $\lambda = -1$ is solvable in the ring $L(\mathbf{R}_+; K_0(x))$ if and only if*

$$1 + \sqrt{\frac{\pi}{2}} \Gamma(\gamma - i\tau)\Gamma(\gamma + i\tau) \sin^{1/2-\gamma} \mu P^{1/2-\gamma}_{-1/2+i\tau}(\cos\mu) \neq 0, \ \tau \in \mathbf{R}, \tag{4.154}$$

where $P^{1/2-\gamma}_{-1/2+i\tau}(\cos\mu)$ is the Legendre function (1.55).

Proof. Actually, inequality (4.154) means condition (4.150) for the Kontorovich-Lebedev transform (2.1) of the function $g(x) = e^{-x\cos\mu} x^{\gamma-1}$. To evaluate this we use formula (1.102), namely for this case we have

$$\int_0^\infty x^{\gamma-1} e^{-x\cos\mu} K_{i\tau}(x) dx$$

$$= \sqrt{\frac{\pi}{2}} \Gamma(\gamma - i\tau)\Gamma(\gamma + i\tau) \sin^{1/2-\gamma} \mu P^{1/2-\gamma}_{-1/2+i\tau}(\cos\mu) \neq 0, \ \tau \in \mathbf{R}, \tag{4.155}$$

which leads us to (4.154). The Corollary 4.2 is proved. •

Corollary 4.3. *Lebedev's equation* (4.134) *is solvable for $\lambda = -2/\pi^2$ in the class $L(\mathbf{R}_+; K_\alpha(x))$, $0 \leq \alpha < 1/2$, and moreover, its unique solution has the form*

$$f(x) = h(x) + \frac{2}{\pi^2} \int_0^\infty \frac{uK_1(u)K_0(x) - xK_1(x)K_0(u)}{x^2 - u^2} u h(u) du. \tag{4.156}$$

Conversely, equation (4.156) *is solvable in the ring $L(\mathbf{R}_+; K_\alpha(x))$, $0 \leq \alpha < 1/2$ and its unique solution has form* (4.134) *in the case $\lambda = -2/\pi^2$.*

Proof. The proof of Corollary 4.3 can be obtained by using the same integral (4.155), when $\mu = 0$, $\gamma = 1/2$. In this case the result is reduced to $K_s[g] = \pi^{3/2}/(\sqrt{2}\cos(\pi s))$, where $g(x) = e^{-x}x^{-1/2}$. Hence write equation (4.152) in the form

$$K_s[f] = \frac{\cos(\pi s)}{1+\cos(\pi s)}K_s[h]$$
$$= (1 + K_s[q])K_s[h], \ |\mu| \leq \alpha, \tag{4.157}$$

where the value of the function $q(x)$ can be obtained by using formula 2.16.33.2 from Prudnikov et al. [2] and $q(x) = -(2/\pi^2)K_0(x)$. The corresponding kernel $\mathcal{K}_q(x,u)$ is calculated from formula (4.3) and by integral 2.16.9.1 in Prudnikov et al. [2]. It leads us to the kernel in the solution (4.156), namely

$$\mathcal{K}_q(x,u) = -\frac{1}{\pi^2 x}\int_0^\infty \exp\left(-\frac{1}{2}\left(\frac{xy}{u}+\frac{xu}{y}+\frac{uy}{x}\right)\right)K_0(y)dy$$
$$= \frac{2}{\pi^2}\frac{u(uK_1(u)K_0(x) - xK_1(x)K_0(u))}{x^2-u^2}. \tag{4.158}$$

The condition $0 \leq \alpha < 1/2$ arises from the convergence of integral (4.39) for $||g||_{L^\alpha} < +\infty$, where $g(x)(x) = e^{-x}x^{-1/2}$. Corollary 4.3 is proved. •

Concerning convolution equation of the first kind like (4.2)

$$\int_0^\infty \mathcal{K}(x,u)f(u)du = h(x) \tag{4.159}$$

briefly note that its solution one can write being based on the factorization equality (4.33) for the Kontorovich-Lebedev transform and its range for corresponding space of functions. Thus if, for example, we look for a solution in the convolution Hilbert space S_q one may take the given function $h(x)$ from S_q as well as the kernel function $g(x)$ (see (4.3)). According to Theorem 4.20 the range $KL(S_q)$ coincides with the Hilbert space H_q. So from (4.159) we have algebraic equation in terms of the Kontorovich-Lebedev transform

$$K_{i\tau}[f]K_{i\tau}[g] = K_{i\tau}[h]. \tag{4.160}$$

Hence

$$K_{i\tau}[f] = \frac{K_{i\tau}[h]}{K_{i\tau}[g]} \tag{4.161}$$

and the solution of equation (4.159) at the space S_q can be written by formula (4.119) if and only if the right-hand side of (4.161) belongs to the space H_q.

4.6 Other convolution constructions

Our object here is to develop ideas from Nguyen Thanh Hai and Yakubovich [1], concerning generalization of the notion of convolution, starting to consider quite general its definition as

$$(f * g)(x) = \mathcal{K}\left[\mathcal{K}_1[f]\mathcal{K}_2[g]\right](x), \tag{4.162}$$

where $\mathcal{K}$, $\mathcal{K}_i$, $i = 1, 2$ are some integral operators, $(f * g)(x)$ is a convolution of two functions $f(x), g(x)$ from a suitable functional space. In the monograph mentioned above we discussed such generalization for the Mellin convolution type transforms, applying the theory of the double Mellin-Barnes type integrals. These results shall be established for the Kontorovich-Lebedev and the Mehler-Fock index transforms, basing on representation formula (4.36) for exponential kernel (4.16). It shall be clear later how to obtain the respective results for other index transforms, appealing to compositions of different integral operators of such type. Precisely speaking, we make use widely composition representations to generalize convolution (4.1) and arrive to the corresponding formula (4.162). Some comments the reader can find in Yakubovich and Luchko [2]. However, here we follow our assumption of L_p-properties of the considered functions.

Before proceeding the key constructions we examine formula (4.36). Indeed, in view of estimate (4.37) we have the following inequality

$$\exp\left(-\frac{1}{2}\left(\frac{xy}{u} + \frac{xu}{y} + \frac{yu}{x}\right)\right) \leq CK_0(x\beta_1)K_0(y\beta_2)K_0(u\beta_3), \tag{4.163}$$

where C is an absolute constant and $\beta_i = \cos\delta_i$, $i = 1, 2, 3$, $\delta_i \in [0, \pi/2)$. Therefore, one can integrate through in the symmetric kernel (4.36) by any variable x, y, u to deduce new convolution kernels. Thus, for example by virtue of formula (1.102), taking there $\alpha = 1/2, x \geq 1$ one can immediately lead to the equality

$$\int_0^\infty K_{i\tau}(u)e^{-xu}\frac{du}{\sqrt{u}} = \frac{\pi\sqrt{\pi}}{\sqrt{2}\cosh\pi\tau}P_{-1/2+i\tau}(x), \tag{4.164}$$

which enables us to construct the convolution for the Mehler-Fock transform being slightly different from (3.1) by simple interchanges of variable and functions. Such convolution was mentioned in Yakubovich and Luchko [2] (see also Yakubovich and Moshinskii [1]). There it was as a corollary of general constructions in the spaces related to the inverse Mellin transform. Nevertheless, here we deduce it directly and shall motivate our discussion for L_p-functions. Namely, multiplying through in (4.36) by the power-exponential expression (see below) and integrating the left- and the right-hand sides by all variables we come to the iterated integral of type

$$\begin{aligned} S(x,y,t) &= \int_0^\infty\int_0^\infty\int_0^\infty \exp\left(-\frac{1}{2}\left(\frac{uv}{w} + \frac{vw}{u} + \frac{wu}{v}\right)\right) \\ &\quad\times\frac{e^{-xu-yv-tw}}{\sqrt{uvw}}dudvdw \\ &= \frac{4}{\pi^2}\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \tau\sinh(\pi\tau)K_{i\tau}(u)K_{i\tau}(v)K_{i\tau}(w) \\ &\quad\times\frac{e^{-xu-yv-tw}}{\sqrt{uvw}}dudvdwd\tau, \quad x, y, t \geq 1. \end{aligned} \tag{4.165}$$

Hence invoking with formula (4.164) and the Fubini theorem that can be performed due to estimate (4.163) it becomes at the right-hand side of (4.165) as

$$S(x,y,t) = \frac{\pi^2\sqrt{\pi}}{\sqrt{2}} \int_0^\infty \tau \frac{\sinh(\pi\tau)}{\cosh^3(\pi\tau)} P_{-1/2+i\tau}(x) P_{-1/2+i\tau}(y) P_{-1/2+i\tau}(t) d\tau. \qquad (4.166)$$

On the other hand, the integral in the left-hand side is already calculated in Yakubovich and Luchko [2] (see formula (16.45)) and we find

$$S(x,y,t) = \sqrt{\frac{2\pi}{D}} \log\left(\frac{x+y+t+1+\sqrt{D}}{x+y+t+1-\sqrt{D}}\right), \ x,y,t \geq 1, \qquad (4.167)$$

where $D = x^2 + y^2 + t^2 - 1 - 2xyt$ and the main values of the square root and the logarithm are taken.

Let us introduce the convolution operator of type

$$(f * g)(x) = \int_1^\infty \int_1^\infty S(x,y,t) f(t) g(y) dt dy, \ x > 1, \qquad (4.168)$$

where $f(x)$, $g(x)$ are defined on $x \in [1, +\infty)$ and the kernel $S(x,y,t)$ is given by formula (4.167). The Mehler-Fock transform in this case corresponds to the operator as

$$P_{-1/2+i\tau}\{f\} = \int_1^\infty P_{-1/2+i\tau}(t) f(t) dt, \ \tau \in \mathbf{R}_+. \qquad (4.169)$$

Hence invoking with representation (4.166), multiplying it through by $f(t)g(y)$ and integrating $[1,+\infty) \times [1,+\infty)$ arrive formally at the following equality

$$(f * g)(x) = \frac{\pi^2\sqrt{\pi}}{\sqrt{2}} \int_0^\infty \tau \frac{\sinh(\pi\tau)}{\cosh^3(\pi\tau)} P_{-1/2+i\tau}(x) P_{-1/2+i\tau}\{f\} P_{-1/2+i\tau}\{g\} d\tau. \qquad (4.170)$$

To motivate it we need to assume some conditions for the existence of the Mehler-Fock transform (4.169) and to discuss the convergence of the integral (4.170). This shall complete our construction of the convolution for the Mehler-Fock transform of kind (4.162), where as it is clear the operators $\mathcal{K}_i$, $i = 1,2$ are the same Mehler-Fock transforms (4.169) and $\mathcal{K}$ means the index operator as

$$(\mathcal{K}f)(x) = \frac{\pi^2\sqrt{\pi}}{\sqrt{2}} \int_0^\infty \tau \frac{\sinh(\pi\tau)}{\cosh^3(\pi\tau)} P_{-1/2+i\tau}(x) f(\tau) d\tau. \qquad (4.171)$$

The desired conditions arise in view of the Hölder inequality (1.8) and the generalized Minkowski inequality (1.10) applying theirs to estimate the Mehler-Fock transform (4.169). Namely, let $f(x)$ be from the space $L_{\nu,p}([1,+\infty))$, $1/2 < \nu < 1$, $p \geq 1$. Hence, recall inequality (1.100) and obtain the estimate as follows

$$\frac{\pi\sqrt{\pi}}{\sqrt{2}\cosh \pi\tau} |P_{-1/2+i\tau}\{f\}|$$

$$= \left| \int_1^\infty f(t) \int_0^\infty K_{i\tau}(u) e^{-tu} \frac{du}{\sqrt{u}} dt \right|$$

$$\leq e^{-\delta\tau}||f||_{L_{\nu,p}([1,+\infty))}$$
$$\times \int_0^\infty K_0(u\cos\delta)u^{\nu-3/2}\left(\int_u^\infty t^{q(1-\nu)-1}e^{-qt}dt\right)^{1/q} du$$
$$\leq q^{\nu-1}\Gamma^{1/q}(q(1-\nu))e^{-\delta\tau}||f||_{L_{\nu,p}([1,+\infty))}\int_0^\infty K_0(u\cos\delta)u^{\nu-3/2}du < \infty \quad (4.172)$$

under conditions $1/2 < \nu < 1, \delta \in (0, \pi/2), q = p/(p-1)$. Thus by virtue of inequality (4.37) which gives the absolute convergence of the integral by τ. Let us formulate the theorem.

Theorem 4.24. *If* $f, g \in L_{\nu,p}([1,+\infty))$, $1/2 < \nu < 1, p \geq 1$, *then the convolution* $(f * g)$ (4.168) *for the Mehler-Fock transform* (4.169) *exists and the Parseval equality* (4.170) *holds. In addition,* $(f * g) \in L_{1-\nu,p}([1,+\infty))$.

Proof. We have to establish only last proposition in this theorem. Indeed, from the above discussions one can estimate integral (4.170) as follows

$$|(f * g)(x)| \leq C\int_0^\infty K_0(\beta u)e^{-xu}\frac{du}{\sqrt{u}},\ \beta \in (0,1),\ x \geq 1, \quad (4.173)$$

where $C > 0$ is an absolute constant. Consequently, it is not difficult to see that according to the generalized Minkowski inequality (1.10) we have

$$||(f * g)||_{L_{1-\nu,p}([1,+\infty))} \leq C\int_0^\infty K_0(\beta u)u^{\nu-3/2}du\left(\int_u^\infty t^{p(1-\nu)-1}e^{-pt}dt\right)^{1/p}$$
$$\leq Cp^{\nu-1}\Gamma^{1/p}(p(1-\nu))\int_0^\infty K_0(\beta u)u^{\nu-3/2}du < \infty \quad (4.174)$$

under the above assumptions on the parameters. This completes the proof of Theorem 4.24. •

We illustrate now the next example of convolution construction (4.162), integrating through only once in formula (4.36). For this purpose multiplying through in (4.36) by the power-exponential function, integrating it and invoking with formula (4.164) we find that

$$\mathcal{S}(x,y,t) = \int_0^\infty \exp\left(-\frac{1}{2}\left(\frac{uy}{t} + \frac{yt}{u} + \frac{ut}{y} - xu\right)\right)\frac{du}{\sqrt{u}}$$
$$= \frac{2\sqrt{2}}{\sqrt{\pi}}\int_0^\infty \tau\tanh(\pi\tau)P_{-1/2+i\tau}(x)K_{i\tau}(y)K_{i\tau}(t)d\tau. \quad (4.175)$$

However, the left-hand side of equality (4.175) one can calculate using formula 2.3.16.1 in Prudnikov et al. [1] and equality (4.132). Thus we obtain

$$\mathcal{S}(x,y,t) = \frac{\sqrt{2\pi yt}\exp\left(-\sqrt{y^2+t^2+2xyt}\right)}{(y^2+t^2+2xyt)^{1/2}}. \quad (4.176)$$

Hence we can consider two different convolution operators

$$(f*g)_1(x)=\sqrt{2\pi}\int_0^\infty\int_0^\infty\frac{\exp\left(-\sqrt{y^2+t^2+2xyt}\right)}{(y^2+t^2+2xyt)^{1/2}}\sqrt{yt}f(y)g(t)dydt$$

$$=\int_0^\infty\int_0^\infty \mathcal{S}(x,y,t)f(y)g(t)dydt,\ x\geq 1, \tag{4.177}$$

$$(f*g)_2(x)=\sqrt{2\pi x}\int_0^\infty\int_1^\infty\frac{\exp\left(-\sqrt{x^2+y^2+2xyt}\right)}{(x^2+y^2+2xyt)^{1/2}}\sqrt{y}f(y)g(t)dydt$$

$$=\int_0^\infty\int_0^\infty \mathcal{S}(t,x,y)f(y)g(t)dydt,\ x>0. \tag{4.178}$$

The following theorems are true.

Theorem 4.25. *Let $f,g\in L_{\nu,p}(\mathbf{R}_+)$, $0<\nu<1/2$, $p\geq 1$. Then convolution* (4.177) *is defined on $[1,+\infty)$, exists for all $x\geq 1$ and belongs to the space $L_{\nu,p}([1,+\infty))$. In addition, the equality like* (4.162) *is valid*

$$(f*g)_1(x)=\frac{2\sqrt{2}}{\sqrt{\pi}}\int_0^\infty \tau\tanh(\pi\tau)P_{-1/2+i\tau}(x)K_{i\tau}[f]K_{i\tau}[g]d\tau, \tag{4.179}$$

where $K_{i\tau}[f]$ is the Kontorovich-Lebedev transform (2.1).

Proof. Indeed, make use estimates (2.11) and (4.37) to observe the uniform convergence by $x\geq 1$ of the integral by index of the Legendre function at the right-hand side of (4.179). Furthermore, we establish the inequality as

$$|(f*g)_1(x)|\leq C||f||_{\nu,p}||g||_{\nu,p}\int_0^\infty K_0(\beta u)e^{-xu}\frac{du}{\sqrt{u}}, \tag{4.180}$$

where norms (1.19) for the functions $f(x),g(x)$ are finite according to the proposition of Lemma 2.1 and conditions of this theorem. Hence, estimate the norm in the space $L_{\nu,p}([1,+\infty))$ of convolution (4.177) by using the generalized Minkowski inequality (1.10) and arrive to the relation as

$$||(f*g)_1||_{L_{\nu,p}([1,+\infty))}\leq C||f||_{\nu,p}||g||_{\nu,p}$$

$$\times\int_0^\infty K_0(\beta u)u^{-\nu-1/2}du\left(\int_u^\infty t^{\nu p-1}e^{-pt}dt\right)^{1/p}$$

$$<Cp^{-\nu}\Gamma^{1/p}(\nu p)||f||_{\nu,p}||g||_{\nu,p}\int_0^\infty K_0(\beta u)u^{-\nu-1/2}du<\infty \tag{4.181}$$

under condition $0<\nu<1/2$ which gives the desired result. Equality (4.179) follows from (4.175) and the Fubini theorem. This ends the proof of Theorem 4.25. •

Theorem 4.26. *Let $f(x)$ be from the space $L_{\nu,p}(\mathbf{R}_+)$ and $g(x)$ be from the space $L_{\nu,p}([1,+\infty))$, where $p\geq 1$, $1/2<\nu<1$. Then convolution* (4.178) *is defined on*

$\mathbf{R}_+$ *and belongs to the space* $L_{\nu,p}(\mathbf{R}_+)$. *Furthermore, the equality of Parseval's type is true*

$$(f * g)_2(x) = \frac{2\sqrt{2}}{\sqrt{\pi}} \int_0^\infty \tau \tanh(\pi\tau) K_{i\tau}(x) K_{i\tau}[f] P_{-1/2+i\tau}[g] d\tau, \tag{4.182}$$

where the Mehler-Fock transform $P_{-1/2+i\tau}[g]$ *is given by formula* (4.169) *and* $K_{i\tau}[f]$ *is the Kontorovich-Lebedev transform* (2.1).

Proof. First confirm the boundedness of the operator $K_{i\tau}[f]$ provided that $f(x) \in L_{\nu,p}(\mathbf{R}_+)$, $1/2 < \nu < 1$, $p \geq 1$ in view of Lemma 2.1 and estimate (2.11). Further, the Mehler-Fock transform (4.169) is a bounded operator by virtue of (4.172). So substituting the value of the kernel $\mathcal{S}(t, x, y)$ by formula (4.175), formally change the order of integration and lead to (4.182). To motivate it by the Fubini theorem we appeal to the estimate of the right-hand side of (4.182). Namely, we find

$$\int_0^\infty \tau \tanh(\pi\tau) |K_{i\tau}(x) K_{i\tau}[f] P_{-1/2+i\tau}[g]| d\tau$$

$$\leq C ||f||_{L_{\nu,p}(\mathbf{R}_+)} ||g||_{L_{\nu,p}([1,+\infty))}$$

$$\times K_0(x \cos\delta) \int_0^\infty \tau e^{(\pi - 3\delta)\tau} d\tau, \ 1/2 < \nu < 1, \tag{4.183}$$

where one can choose $\delta \in (\pi/3, \pi/2)$ according to inequality (1.100) and the integral by τ becomes convergent. Hence we deduced equality (4.182) and evidently can conclude by virtue of (4.183) that $(f * g)_2(x) \in L_{\nu,p}(\mathbf{R}_+)$, $1/2 < \nu < 1$. This completes the proof of Theorem 4.26. •

Let us consider at the end of this chapter one convolution integral connected with the inverse Kontorovich-Lebedev transform (see, for instance expansion (1.231)). One can introduce the following index transform

$$\mathcal{K}^{-1}[f](x) = \int_0^\infty K_{i\tau}(x) f(\tau) d\tau, \tag{4.184}$$

where $x > 0$ and the integration is realized here by the index of the Macdonald function (1.98). Inequality (1.100) and the Hölder inequality (1.8) immediately give us the uniform estimate of operator (4.184). Indeed, if $f(\tau) \in L_{\nu,p}(\mathbf{R}_+)$ with $\nu < 1$, $p \geq 1$, then we have

$$|\mathcal{K}^{-1}[f](x)| \leq K_0(x\cos\delta) \int_0^\infty e^{-\delta\tau} |f(\tau)| d\tau$$

$$\leq K_0(x\cos\delta) ||f||_{\nu,p} \left(\int_0^\infty e^{-q\delta\tau} \tau^{q(1-\nu)-1} d\tau \right)^{1/q}, \ \delta \in (0, \pi/2), q = p/(p-1). \tag{4.185}$$

Hence it is easily to conclude that $\mathcal{K}^{-1}[f](x) \in L_{\nu,p}(\mathbf{R}_+)$, $0 < \nu < 1$, combining with the above condition on the parameter ν.

Take now formula 2.16.46.6 from Prudnikov et al. [2] that can be written in terms of notations of Chapter 1 (see, for example (1.163)). Namely, we have the integral of the product of three different Macdonald functions

$$\int_0^\infty x^{\alpha-1}K_{i\beta}(x)K_{i\xi}(x)K_{i\tau}(x)dx$$
$$= \Re_{i\tau}\left[\frac{\Gamma(i\tau)}{\Gamma(\alpha-i\tau)}\Gamma\left(\frac{\alpha+i\beta+i\xi-i\tau}{2}\right)\Gamma\left(\frac{\alpha-i\beta+i\xi-i\tau}{2}\right)\right.$$
$$\times\Gamma\left(\frac{\alpha+i\beta-i\xi-i\tau}{2}\right)\Gamma\left(\frac{\alpha-i\beta-i\xi-i\tau}{2}\right)$$
$$\times{}_4F_3\left(\frac{\alpha+i\beta+i\xi-i\tau}{2},\frac{\alpha-i\beta+i\xi-i\tau}{2},\frac{\alpha+i\beta-i\xi-i\tau}{2},\frac{\alpha-i\beta-i\xi-i\tau}{2};\right.$$
$$\left.\left.1-i\tau,\frac{\alpha-i\tau}{2},\frac{1+\alpha-i\tau}{2};\frac{1}{4}\right)\right], \tag{4.186}$$

which contains the hypergeometric function ${}_4F_3(a_1,a_2,a_3,a_4;b_1,b_2,b_3;z)$ (1.45) at the point $z=1/4$ and it depends upon the parameters $\beta,\xi,\tau\in\mathbf{R}$ and $\Re\alpha>0$. Denoting the right-hand side of (4.186) by $S^{\alpha}_{i\beta,i\xi,i\tau}$ define the convolution operator

$$(f*g)(\tau)=\int_0^\infty S^{\alpha}_{i\beta,i\xi,i\tau}f(\beta)g(\xi)d\beta d\xi. \tag{4.187}$$

Theorem 4.27. *Let $f,g\in L_{\nu,p}(\mathbf{R}_+)$, $0<\nu<1$, $p\geq 1$. Then convolution* (4.187) *belongs to the same space $L_{\nu,p}(\mathbf{R}_+)$ and the following representation is true for all $\tau\in\mathbf{R}_+$, namely*

$$(f*g)(\tau)=\int_0^\infty x^{\alpha-1}K_{i\tau}(x)\mathcal{K}^{-1}[f](x)\mathcal{K}^{-1}[g](x)dx, \tag{4.188}$$

where the operator $\mathcal{K}^{-1}$ is given by formula (4.184).

Proof. It is clear, that this proof is similar to the proof of the above theorems. By virtue of estimate (4.185) we obtain for the kernel $S^{\alpha}_{i\beta,i\xi,i\tau}$ the inequality

$$\left|S^{\alpha}_{i\beta,i\xi,i\tau}\right|\leq e^{-\delta(\beta+\xi+\tau)}$$
$$\times\int_0^\infty x^{\alpha-1}K_0(x\cos\delta)K_0(x\cos\delta)K_0(x\cos\delta)dx<\infty,\ \Re\alpha>0, \tag{4.189}$$

under the same assumption for the parameter δ. Consequently, the absolute and uniform convergence of integral (4.186) is established. Multiplying this integral through by $f(\beta)g(\xi)$ and integrating it by measure $(\mathbf{R}_+\times\mathbf{R}_+;d\beta d\xi)$ with the aid of the Fubini theorem arrive to the desired formula (4.184). The belonging to the space $L_{\nu,p}(\mathbf{R}_+)$, $0<\nu<1$ of the convolution $(f*g)(\tau)$ follows from the uniform estimate

$$|(f*g)(\tau)|\leq Ce^{-\delta\tau}, \tag{4.190}$$

that can be deduced from inequality (4.189). Theorem 4.27 is proved. •

Chapter 5

General Index Transforms

In this chapter we study general index transforms, basing on the theory of the Kontorovich-Lebedev transform from Chapter 2. We need to use very important results from the Mellin transform L_p-theory in Chapter 1. In fact, as we already know the hypergeometric structure of the index kernels enables to apply the Mellin-Parseval equality like (1.214) and investigate the questions of the mapping and invertibility for the index integral operators. Such objects are slightly exhibited in Section 2.5, where we introduced enough general index kernel (2.128). Here we develop this approach and give several examples of index transforms with special functions as the kernels being listed, for instance, in Chapter 1 as particular cases of Meijer's G-function.

However, our purpose at the beginning of this chapter to apply rigorous results from the theory of Banach spaces of analytic functions in the complex domain to extend the respective properties for key index transforms as the Kontorovich-Lebedev and the Mehler-Fock index operators. Namely, their composition structure allows to recall familiar theorems of the identification of images of the Laplace and the Fourier transforms in the Hardy type spaces (see Duren [1]) and to prove the analogs of *the Paley-Wiener theorem* for the index transforms in Hilbert spaces. Analytic properties of the mentioned transforms have been studied in Chapter 2. Here using elements of the theory of the Hardy spaces we extend the Kontorovich-Lebedev transform and the Mehler-Fock transform to the variable from a complex domain. Moreover, we draw a parallel with the classical Hardy spaces and their Hilbert's analogs as the Bergman-Selberg space and the Szegö space. It allows us to identify the image of the mentioned index transforms like for the Fourier and the Laplace transforms in classical cases. We refer the reader to the questions that are considered in detail for the Fourier transform, the Laplace transform and some other familiar integral transforms with using the theory of *reproducing kernel Hilbert spaces* by Aronszajn [1], Bergman [1], Saitoh [3]. Concerning the theory of the Hardy spaces see, for instance, Duren [1].

5.1 The Kontorovich-Lebedev transform in a complex domain. Analogs of the Hardy type spaces and the Paley-Wiener theorem

At first to introduce the Kontorovich-Lebedev transform which we shall investigate in this section let us attract the attention of the reader to the classical elements of the theory of the Hardy spaces and their Hilbert analogs.

As is known the classical *Hardy type H_p spaces* are defined for $0 < p < \infty$ to consist of those functions F, holomorphic (analytic) in the right half-plane, with the property that $\mu_p(F, x)$ is bounded for $x > 0$, where

$$\mu_p(F, x) = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} |F(x+iy)|^p dy\right)^{1/p}. \tag{5.1}$$

There is the well-known connection between these spaces and Laplace transforms $F(z) = [Lf](z)$ (1.215) of a complex variable $z, \Re z > 0$ of functions $f \in L_p(\mathbf{R}_+), 1 < p \leq 2$. However, if we consider the Fourier transform like (1.191)

$$F(z) = \int_0^{\infty} f(t)e^{itz}dt, \tag{5.2}$$

where the variable z lies at the upper half-plane and $f \in L_p(\mathbf{R}_+), 1 < p \leq 2$, then it corresponds to the case of the H_p spaces with

$$\mu_p(F, y) = \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} |F(x+iy)|^p dx\right)^{1/p} \tag{5.3}$$

and $y > 0$. As is evident the case $p = 2$ related to the Hilbert spaces. More precisely, one can define here *the Szegö spaces*

$$||F|| = \sup_{x>0}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} |F(x+iy)|^2 dy\right)^{1/2}, \tag{5.4}$$

$$||F|| = \sup_{y>0}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} |F(x+iy)|^2 dx\right)^{1/2}. \tag{5.5}$$

Thus equalities (5.4)-(5.5) determine the Hilbert spaces H with the respective inner product $\langle\ ,\ \rangle_H$. Following the general theory of reproducing kernels from Aronszajn [1] one can describe the Hilbert space H admitting some reproducing kernel $H(z, \bar{u})$, which is the function of two complex variables z, u from the respective domain and the bar means usual complex conjugation. Let us demonstrate this application in the case of the Fourier and the Laplace transform.

Turn now to the Laplace transform (1.215) in slightly modified form as

$$F(z) \equiv [Lf](z) = \int_0^{\infty} e^{-zt} f(t)t^{2\nu-1}dt, \tag{5.6}$$

where z belongs to the right half-plane $\Re z > 0$ and $f(t) \in L_{\nu,2}(\mathbf{R}_+), \nu > 0$. Then according to the Hölder inequality (1.21) and invoking with (1.19) estimate the absolute value of the Laplace transform (5.6) as follows

$$|F(z)| \leq ||f||_{\nu,2} \left(\int_0^\infty e^{-2\Re z t} t^{2\nu-1} dt \right)^{1/2} < \infty, \tag{5.7}$$

when $\nu > 0$ and it defines the analytic function in the open right half-plane $\{\mathbf{R}^+ : \Re z > 0\}$ due to usual test of an absolute and uniform convergence of integrals with a complex parameter and Morera's theorem. Call the identity

$$H(z, \bar{u}) = \int_0^\infty e^{-(z+\bar{u})t} t^{2\nu-1} dt = \frac{\Gamma(2\nu)}{(z + \bar{u})^{2\nu}} \tag{5.8}$$

for $\Re z$, $\Re u > 0$, $\nu > 0$. This function is identified with the uniquely determined reproducing kernel by the corresponding Hilbert space related to the Laplace integral (5.6). Namely, as is known identity (5.8) implies that the image of the space $L_{\nu,2}(\mathbf{R}_+)$ under the Laplace transform $[Lf](z)$ coincides with the *Bergman-Selberg* Hilbert space H_ν (see Saitoh [4]) admitting the reproducing kernel (5.8) $H(z, \bar{u})$. Namely, the Bergman-Selberg space H_ν consists of the functions $F(z)$ analytic in the right half-plane $z \in \mathbf{R}^+$, $\Re z > 0$ with finite norms

$$||F||_\nu = \frac{2^{\nu-1}}{\sqrt{\pi \Gamma(2\nu - 1)}} \left(\int\int_{\mathbf{R}^+} |F(z)|^2 x^{2\nu-2} dx dy \right)^{1/2}, \; z = x + iy. \tag{5.9}$$

Thus, since the set of functions $\{e^{-zt}, \; z \in \mathbf{R}+\}$ is complete in $L_{\nu,2}(\mathbf{R}_+)$ we have from (5.8) that any member $F(z)$ in H_ν is expressible in form (5.6) for uniquely determined function $f \in L_{\nu,2}(\mathbf{R}_+)$ satisfying

$$||f||_{\nu,2} = \left(\int_0^\infty |f(t)|^2 t^{2\nu-1} dt \right)^{1/2} < \infty. \tag{5.10}$$

Prove now directly that for $\nu > 1/2$ the Parseval-Plancherel identity

$$||F||_\nu^2 = \int_0^\infty |f(t)|^2 t^{2\nu-1} dt \tag{5.11}$$

is true. Indeed, by definition of the norm (5.9) the double integral at its right hand-side by $\mathbf{R}^+$ can be calculated invoking with the Fubini theorem and usual Parseval's equality for the Fourier transform (1.191) (see, for instance Titchmarsh [1]). First we find that for any $F \in H_\nu$ the Fubini theorem implies that integrals

$$\int_{-\infty}^\infty |F(x + iy)|^2 dy \tag{5.12}$$

absolutely converge for almost all $x \in \mathbf{R}_+$. Invoking with formula (5.6) and substituting it in integral (5.12) use the Parseval identity for the Fourier transform and obtain that

$$\int_{-\infty}^\infty |F(x + iy)|^2 dy = 2\pi \int_0^\infty e^{-2xt} |f(t)|^2 t^{4\nu-2} dt. \tag{5.13}$$

Hence integrating through by $x \in \mathbf{R}_+$ in (5.13) after multiplying through there by $x^{2\nu-2}$ and using Fubini's theorem, we calculate the inner integral by formula (1.22) for $\nu > 1/2$ and arrive to the equality

$$\int\int_{\mathbf{R}^+} |F(x+iy)|^2 x^{2\nu-2} dxdy$$
$$= 4^{1-\nu}\pi\Gamma(2\nu-1)\int_0^{\infty} |f(t)|^2 t^{2\nu-1} dt. \tag{5.14}$$

This leads to (5.11) according to (5.9). The limit case $\nu = 1/2$ comprises the famous Szegö space with norm (5.4) and the respective reproducing kernel (5.8) corresponds to the Laplace transform (1.215) of the complex variable z of a function $f \in L_2(\mathbf{R}_+)$. Furthermore, $F(z) = [Lf](z)$ is analytic function in the right half-plane, and as is well known in the theory of analytic functions, that $F(x+iy)$ satisfying (5.4) has a non-tangential boundary values almost everywhere on the imaginary line ($x = 0$). It gives the familiar *Paley-Wiener theorem* (see Akhiezer [1], Duren [1]).

Theorem 5.1. *A function $F(z)$ belongs to the Szegö space with norm* (5.4) *if and only if it has the form* (1.215) *for some $f \in L_2(\mathbf{R}_+)$.*

Next, one can spread the Bergman-Selberg space (5.9) on the general exponent p, where $1 < p < 2$, $q = p/(p-1)$, considering the functions $F(z)$ analytic in the right half-plane $z \in \mathbf{R}^+$, $\Re z > 0$ with finite norms

$$||F||_{\mathcal{B}}^q = \frac{(2\pi)^{-1+(q-p)/2} p^{1+p(\nu-1)}}{\Gamma(p(\nu-1)+1)} \int\int_{\mathbf{R}^+} |F(z)|^q x^{p(\nu-1)} dxdy, \ z = x+iy. \tag{5.15}$$

In this case the Laplace transform (5.6) can be estimated by inequality (1.21) as follows

$$|F(z)| \leq ||f||_{\nu,p}\left(\int_0^{\infty} e^{-q\Re z t} t^{q\nu-1} dt\right)^{1/q} < \infty, \ q = p/(p-1) \tag{5.16}$$

under condition $\nu > 0$ and $f \in L_{\nu,p}(\mathbf{R}_+)$. Further, make use Theorem 1.14 and inequality (1.193). It gives us the estimate for the Laplace transform $F(z)$ (5.6) as

$$\int_{-\infty}^{\infty} |F(x+iy)|^q dy \leq (2\pi)^{1+(p-q)/2} \int_0^{\infty} e^{-pxt} t^{p(2\nu-1)} |f(t)|^p dt. \tag{5.17}$$

Hence in the same manner multiplying through by $x^{p(\nu-1)}$ in inequality (5.17) and integrating then by $x \in \mathbf{R}_+$ immediately obtain that

$$\int\int_{\mathbf{R}^+} |F(x+iy)|^q x^{p(\nu-1)} dxdy$$
$$\leq (2\pi)^{1+(p-q)/2} p^{p(1-\nu)-1}\Gamma(p(\nu-1)+1)\int_0^{\infty} |f(t)|^p t^{p\nu-1} dt \tag{5.18}$$

provided that $f \in L_{\nu,p}(\mathbf{R}_+)$, $\nu > 1/q$. Thus according to the definition of norm (5.15) we finally arrive to the estimate

$$||F||_{\mathcal{B}}^q \leq ||f||_{\nu,p}^p. \tag{5.19}$$

Similar realization of the Hilbert space admitting the reproducing kernel can be given for the Fourier transform (5.2) of L_2-functions $f(t)$. In this case we obtain analytic functions $F(z)$ in the upper half-plane $U \in \mathbf{C}$. As is known the corresponding reproducing kernel one can deduce from the identity

$$H(z, \bar{u}) = \int_0^\infty e^{itz} e^{-it\bar{u}} dt = \frac{i}{z - \bar{u}}, \tag{5.20}$$

where $z, u \in U$ and the set of functions $\{e^{-itz}, z \in U\}$ is complete in $L_2(\mathbf{R}_+)$. Some details the reader can find in Duren [1]. It is naturally to expose here the respective Paley-Wiener theorem.

Theorem 5.2. *A function $F(z)$ being analytic in the upper half-plane U of the complex plane $\mathbf{C}$ belongs to space* (5.5) *if and only if it has form* (5.2) *for some $f \in L_2(\mathbf{R}_+)$.*

Let us begin now to study the Kontorovich-Lebedev transforms of several types in a complex domain drawing a parallel with the results examined above concerning classical Fourier and Laplace transforms. We already introduced in Chapter 4 the Kontorovich-Lebedev transform $\mathcal{K}^{-1}[f](x)$ by formula (4.184). Here set by $\mathcal{K}^{-1}[f](z)$ the index transform of type

$$\mathcal{K}^{-1}[f](z) \equiv F(z) = \int_0^\infty \tau \tanh(\pi\tau) K_{i\tau}(z) f(\tau) d\tau, \tag{5.21}$$

where the variable z as we show below is reasonable to take from the right half-plane $\{\mathbf{R}^+ : \Re z > 0\}$. We shall define the corresponding Hilbert spaces for $f(\tau)$ and its image later after investigation the analytic properties of the kernel $K_{i\tau}(z)$. For this appeal to integral representation (1.98), which one can write in the form

$$K_{i\tau}(z) = \int_0^\infty e^{-z\cosh v} \cos(\tau v) dv. \tag{5.22}$$

Hence we find easily by estimate

$$|K_{i\tau}(z)| \leq \int_0^\infty e^{-\Re z \cosh v} dv = K_0(\Re z), \tag{5.23}$$

where as is evident $\Re z > 0$ and consequently, the function $K_{i\tau}(z)$ is analytic there by variable z (we mean in the open right half-plane $\mathbf{R}^+$). Let us consider the Hilbert weighted space $L_2(\mathbf{R}_+; \tau \tanh(\pi\tau))$ with the norm

$$||f||_{L_2(\mathbf{R}_+;\tau\tanh(\pi\tau))} = \left(\int_0^\infty \tau \tanh(\pi\tau) |f(\tau)|^2 d\tau\right)^{1/2} \tag{5.24}$$

and the inner product

$$\langle f, g \rangle = \int_0^\infty \tau \tanh(\pi\tau) f(\tau) \overline{g(\tau)} d\tau. \tag{5.25}$$

One can show that the set of functions $\{K_{i\tau}(z),\ z \in \mathbf{R}^+\}$ is complete in the Hilbert space $L_2(\mathbf{R}_+; \tau \tanh(\pi\tau))$. It means that from equality

$$\mathcal{K}^{-1}[f](z) \equiv 0 \tag{5.26}$$

it follows that $f = 0$ almost everywhere on $\mathbf{R}_+$. Indeed, for each $z \in \mathbf{R}^+$ the kernel $K_{i\tau}(z) \in L_2(\mathbf{R}_+; \tau \tanh(\pi\tau))$ in view of the convergent integral 2.16.52.10 from Prudnikov et al. [2]

$$\int_0^\infty \tau \tanh(\pi\tau)|K_{i\tau}(z)|^2 d\tau = \frac{\pi}{4}\frac{|z|}{x}e^{-2x}. \tag{5.27}$$

Further, for the real variable $z = x > 0$ we apply through in (5.26) the cosine Fourier transform (1.197) (see also (4.126)-(4.128)). After changing the order of integration by the Fubini theorem owing to an absolute convergence of the iterated integral calculate the inner integral by formula 2.16.14.1 in Prudnikov et al. [2] and arrive to the equality

$$\int_0^\infty \frac{\tau \sinh(\pi\tau/2)}{\cosh(\pi\tau)} f(\tau) \cos\left(\tau \log\left(x + \sqrt{x^2+1}\right)\right) d\tau \equiv 0. \tag{5.28}$$

Hence since the integrand function in (5.28) $\frac{\tau \sinh(\pi\tau/2)}{\cosh(\pi\tau)} f(\tau) \in L_1(\mathbf{R}_+)$ then we have there $f(\tau) = 0$ almost everywhere on $\mathbf{R}_+$. Consequently, it remains in (5.26) for all $z \in \mathbf{R}^+$ according to the uniqueness theorem for analytic functions.

Thus one can introduce the reproducing kernel Hilbert space $H_{\mathcal{K}^{-1}}$, which denote the range of the Kontorovich-Lebedev transform $F(z)$ (5.21) for $L_2(\mathbf{R}_+; \tau \tanh(\pi\tau))$ and the inner product in $H_{\mathcal{K}^{-1}}$ induced from the norm (Saitoh [3])

$$||F||_{H_{\mathcal{K}^{-1}}} = \inf\{||f||_{L_2(\mathbf{R}_+; \tau \tanh(\pi\tau))}; F(z) = \mathcal{K}^{-1}[f](z)\}. \tag{5.29}$$

Furthermore, the space $H_{\mathcal{K}^{-1}}$ admits the reproducing kernel by the following index integral

$$\mathcal{K}(z, \bar{u}) = \int_0^\infty \tau \tanh(\pi\tau) K_{i\tau}(z) K_{i\tau}(\bar{u}) d\tau, \tag{5.30}$$

where the points $z, \bar{u}$ are taken from the right half-plane $\mathbf{R}^+$. Invoking with formula 2.16.52.10 from Prudnikov et al. [2] we find that

$$\mathcal{K}(z, \bar{u}) = \frac{\pi}{2}\frac{\sqrt{z\bar{u}}}{z + \bar{u}}e^{-z-\bar{u}}, \tag{5.31}$$

where we choose the main value of the square root. Of course, for any fixed u the function $\mathcal{K}(z, \bar{u})$ belongs to the space $H_{\mathcal{K}^{-1}}$ and moreover, we have for any $F \in H_{\mathcal{K}^{-1}}$ the reproducing property

$$F(u) = \langle F(z), \mathcal{K}(z, \bar{u}) \rangle_{H_{\mathcal{K}^{-1}}}. \tag{5.32}$$

In addition, since the set of functions $\{K_{i\tau}(z)\}$ is complete in the space $L_2(\mathbf{R}_+; \tau \tanh(\pi\tau))$, then the Kontorovich-Lebedev transform (5.21) is an isometry from $L_2(\mathbf{R}_+; \tau\tanh(\pi\tau))$ onto $H_{\mathcal{K}^{-1}}$. Meanwhile, $F(z) = \mathcal{K}^{-1}[f](z)$ for each $z \in \mathbf{R}^+$ is a bounded functional due to the estimate

$$|\mathcal{K}^{-1}[f](z)| \le \int_0^\infty \tau\tanh(\pi\tau)|K_{i\tau}(z)f(\tau)|d\tau$$

$$\le ||f||_{L_2(\mathbf{R}_+;\tau\tanh(\pi\tau))}\left(\int_0^\infty \tau\tanh(\pi\tau)|K_{i\tau}(z)|^2 d\tau\right)^{1/2}$$

$$= \frac{\sqrt{\pi}}{2}\frac{\sqrt{|z|}}{x}e^{-x}||f||_{L_2(\mathbf{R}_+;\tau\tanh(\pi\tau))},\ z = x+iy \tag{5.33}$$

which defines the norm (5.29) in the reproducing Hilbert space $H_{\mathcal{K}^{-1}}$.

However, on the other hand, one can obtain the realization of the space $H_{\mathcal{K}^{-1}}$ by means of the representation of the kernel $\mathcal{K}(z,\bar{u})$ through the integral

$$\mathcal{K}(z,\bar{u}) = \frac{\pi}{2}\frac{\sqrt{z\bar{u}}}{z+\bar{u}}e^{-z-\bar{u}}$$

$$= \frac{\pi}{2}\sqrt{z\bar{u}}\int_1^\infty e^{-zt}e^{-\bar{u}t}dt. \tag{5.34}$$

Consequently, since the set of functions $\{\frac{\pi}{2}\sqrt{z}e^{-zt}, z \in \mathbf{R}^+\}$ is complete in $L_2([1,\infty))$, then the operator

$$F(z) = \frac{\pi}{2}\sqrt{z}\int_1^\infty e^{-zt}h(t)dt \tag{5.35}$$

is an isometry from $L_2([1,\infty))$ onto reproducing kernel Hilbert space $H_{\mathcal{K}^{-1}}$. Hence, by virtue of the above results for the Laplace transform it implies that norm (5.29) can be represented through the equality

$$||F||_{H_{\mathcal{K}^{-1}}} = \inf\{||h||_{L_2([1,\infty))}\} \tag{5.36}$$

and the Kontorovich-Lebedev transform (5.21) is expressible in form (5.35) for some uniquely determined function $h \in L_2([1,\infty))$. Hence making use the definition of the Szegö norm (5.4) we immediately obtain the isometrical identity of type

$$||F||^2_{H_{\mathcal{K}^{-1}}} = \int_1^\infty |h(t)|^2 dt, \tag{5.37}$$

where we mean here that $H_{\mathcal{K}^{-1}}$ composed of all analytic functions $F(z)$ in the right half-plane $\mathbf{R}^+$ with finite norms

$$||F||^2_{H_{\mathcal{K}^{-1}}} = \frac{2}{\pi^3}\sup_{x>0}\int_{-\infty}^\infty \frac{|F(x+iy)|^2}{\sqrt{x^2+y^2}}dy. \tag{5.38}$$

Thus one can conclude that we already established the analog of the Paley-Wiener theorem for the Kontorovich-Lebedev transform (5.21) as the corollary of the respective Theorem 5.1 for the Laplace transform.

Theorem 5.3. *The Kontorovich-Lebedev transform $F(z)$ (5.21) of the function $f \in L_2(\mathbf{R}_+; \tau\tanh(\pi\tau))$ being analytic in the right half-plane $\mathbf{R}^+$ of the complex* *plane* $\mathbf{C}$ *belongs to the reproducing kernel Hilbert space* (5.38) *if and only if it can be represented in the form* (5.35) *for some uniquely determined function* $h \in L_2([1,\infty))$.

One can construct the generalization of the Hilbert space (5.38) on the exponent $1 < p < 2$, using the same method as above for the Laplace transform. However, we choose an arbitrary weighted function, which is different from the power function in general case.

Theorem 5.4. *If the function* $h(t)$ *from representation* (5.35) *belongs to the space* $L_p([1,\infty))$ *with* $1 < p < 2$, *then the Kontorovich-Lebedev transform* $F(z)$ *defined by formula* (5.21) *can be estimated as follows*

$$||F||^q_{\mathcal{K}^{-1}} \leq ||h||^p_{L_p([1,\infty))}, \quad q = p/(p-1), \tag{5.39}$$

and consequently, it belongs to the Banach space of analytic functions $F(z)$ *in the right half-plane* $\mathbf{R}^+$ *equipped with the norm* $||\ ||_{\mathcal{K}^{-1}}$, *namely*

$$||F||^q_{\mathcal{K}^{-1}} = \frac{\pi^{-1-(p+q)/2}2^{-1+(3q-p)/2}}{M_{\rho,p}} \int\int_{\mathbf{R}^+} \frac{\rho(x)|F(z)|^q}{|z|^{q/2}} dxdy, \tag{5.40}$$

where $z = x + iy$ *and the constant* $M_{\rho,p}$ *is defined by formula*

$$M_{\rho,p} = \int_0^\infty \rho(t) e^{-pt} dt < \infty \tag{5.41}$$

for some positive weighted function $\rho(x), x \in \mathbf{R}_+$.

Proof. Indeed, starting from representation (5.35) recall inequality (1.193) from Theorem 1.14 that becomes as

$$\pi^{-1-(p+q)/2}2^{-1+(3q-p)/2} \int_{-\infty}^\infty \frac{|F(x+iy)|^q}{(x^2+y^2)^{q/4}} dy \leq \int_1^\infty e^{-pxt} |h(t)|^p dt. \tag{5.42}$$

Hence, multiplying through in (5.42) by $\rho(x)$ and integrating after on $\mathbf{R}_+$ we perform to change the order of integration at the right-hand side of obtained inequality by the Fubini theorem and arrive to the following estimate

$$\pi^{-1-(p+q)/2}2^{-1+(3q-p)/2} \int\int_{\mathbf{R}^+} \frac{\rho(x)|F(z)|^q}{|z|^{q/2}} dxdy \leq \int_1^\infty |h(t)|^p$$

$$\times \int_0^\infty \rho(x) e^{-pxt} dxdt \leq M_{\rho,p} ||h||^p_{L_p([1,\infty))}. \tag{5.43}$$

Thus invoking with (5.40) we easily lead to estimate (5.33). This ends the proof of Theorem 5.4. •

Let us return now to the Kontorovich-Lebedev transform (2.84) in the form $K_z[f]$, $z = x + iy$ to establish similar properties in Banach spaces of analytic functions. As is known by virtue of Lemma 2.5 the K-L transform $F(z) = K_z[f]$ is analytic function at the symmetric strip $\{\mathcal{S}_{1/2} : |\Re z| < 1/2\}$ for functions $f \in L_2(\mathbf{R}_+)$. Consequently, according to the classical theory of conformal mappings one can apply the theory of H^p spaces by mapping the strip $\mathcal{S}_{1/2}$ onto the unit disk. Moreover, we appeal to Duren [1] to obtain the boundary values, the factorization, integral representations and a harmonic majorant of the analytic functions $F(z)$ in $\mathcal{S}_{1/2}$. As we established the function $K_z[f]$ is analytic in this strip too. Furthermore, due to the evenness of the Macdonald function by index we have the subspace of such functions in this strip, when $F(z) = F(-z)$.

Our final purpose now is to prove the Paley-Wiener theorem for the Kontorovich-Lebedev transform (2.84) and to correspond the Hilbert spaces, which can realize the identification of the image $K_z[f]$ of L_2-function $f(x)$. As we shall see we need to recall the results obtained above in Chapter 2 concerning the Hilbert convolution spaces of functions and Corollary 2.2 that contains the composition representation of the index transform (2.84) by means of the Mellin and the Laplace transforms of special argument.

First make use Corollary 2.2. However, let us take the function f being from the convolution Hilbert space like (2.110), where we allow to set $\sigma = 1/2$. Thus from equality (2.95) one can define the inner product of this Hilbert space, denoting it as $H_{\mathcal{K}}$ by formula

$$\langle f, g\rangle_{H_{\mathcal{K}}} = \frac{1}{2}\int_0^\infty \int_0^\infty K_1(u+v) f(u)\overline{g(v)}dudv \tag{5.44}$$

and equipped with the norm $||f||_{H_{\mathcal{K}}} = \sqrt{\langle f, f\rangle}$.

The following lemma proves that the space $H_{\mathcal{K}}$ is a subspace of $L_2(\mathbf{R}_+)$.

Lemma 5.1. *The space $H_{\mathcal{K}}$ is contained in $L_2(\mathbf{R}_+)$.*

Proof. To prove this fact we appeal to the familiar integral analog of the Hilbert inequality (see, for instance, Duren [1])

$$\int_0^\infty \int_0^\infty \frac{|f(x)g(y)|}{x+y} dxdy \le \pi ||f||_2 ||g||_2. \tag{5.45}$$

Hence we find,

$$\begin{aligned} ||f||^2_{H_{\mathcal{K}}} &= \frac{1}{2}\int_0^\infty \int_0^\infty (u+v)K_1(u+v)\frac{f(u)\overline{f(v)}}{u+v}dudv \\ &\le C\int_0^\infty \int_0^\infty \frac{|f(x)f(y)|}{x+y}dxdy \le \pi C||f||^2_2, \end{aligned} \tag{5.46}$$

where $xK_1(x) < C$, $x > 0$ in view of the asymptotic behavior of the Macdonald function (1.96)-(1.97). This completes the proof of Lemma 5.1. •

This lemma enables to conclude that for functions $f \in H_{\mathcal{K}}$ composition (2.88) is true. Moreover, it remains true for the set of Hilbert spaces $H_{\mathcal{K},\sigma}, |\sigma| < 1/2$ equipped with the norm

$$||f||^2_{H_{\mathcal{K},\sigma}} = \frac{1}{2}\int_0^\infty \int_0^\infty K_{2\sigma}(u+v) f(u)\overline{f(v)}dudv \tag{5.47}$$

and evident embedding

$$H_{\mathcal{K},\sigma_1} \subset H_{\mathcal{K},\sigma_2} \tag{5.48}$$

if $\sigma_1 < \sigma_2$. Without loss of generality one can consider $0 < \sigma < 1/2$, accounting the evenness of the Macdonald function by its index. To be sure in embedding (5.48) use formula (2.120) for the Macdonald function. Further, invoking with the Parseval relation for the Mellin transform (1.203) like (1.214) (see Titchmarsh [1]) it follows that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |K_{x+iy}[f]|^2 dy = \frac{1}{4}\int_0^{\infty} |[Lf](h(t))|^2 t^{2x-1} dt$$

$$= \frac{1}{4}\int_0^{\infty} t^{2x-1} dt \int_0^{\infty} e^{-h(t)u} f(u) du \int_0^{\infty} e^{-h(t)v}\overline{f(v)} dv, \tag{5.49}$$

where the function $h(t)$ is defined by integral (2.83). Therefore, we shall immediately apply it to change the order of integration by the Fubini theorem and by the same motivation as in Theorem 2.5. Hence by virtue of (5.47) it implies the equality

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |K_{x+iy}[f]|^2 dy = ||f||^2_{H_{\mathcal{K},x}}. \tag{5.50}$$

Taking through the operation of supremum by $x \in (0, 1/2)$ in equality (5.50) by using embedding (5.48) arrive to the relation

$$\sup_{0<x<1/2} \frac{1}{2\pi}\int_{-\infty}^{\infty} |K_{x+iy}[f]|^2 dy = ||f||^2_{H_{\mathcal{K}}}. \tag{5.51}$$

Thus we establish the analog of the Szegö space (5.5) for the Kontorovich-Lebedev transform (2.84) and isometrical identity (5.51). To formulate the respective Paley-Wiener theorem for the Kontorovich-Lebedev transform we need to prove converse fact of the representation of an arbitrary complex-valued analytic function $F(z)$ at the strip $\mathcal{S}_{1/2}$ such that $F(z) = F(-z)$ through the Kontorovich-Lebedev integral (2.84) of the function $f \in H_{\mathcal{K}}$. To confirm the identity (5.51) by embedding (5.48) one can use the representation of the Macdonald function as

$$K_{2x}(u+v) = \int_0^{\infty} e^{-(u+v)\cosh t} \cosh(2xt) dt, \tag{5.52}$$

substituting it in the right-hand side of (5.50) and changing the order of integration by the Fubini theorem. Precisely, we have

$$||f||^2_{H_{\mathcal{K},x}} = \frac{1}{2}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} e^{-(u+v)\cosh t} \cosh(2xt) f(u)\overline{f(v)} du dv dt$$

$$= \frac{1}{2}\int_0^{\infty} \cosh(2xt) \left|\int_0^{\infty} e^{-u\cosh t} f(u) du\right|^2 dt$$

$$\leq \frac{1}{2}\int_0^{\infty} \cosh t \left|\int_0^{\infty} e^{-u\cosh t} f(u) du\right|^2 dt = ||f||^2_{H_{\mathcal{K}}}. \tag{5.53}$$

Return now to the case, when we have an arbitrary function $F(z) = F(-z)$ being analytic at the strip $|\Re z| < 1/2$, which belongs to the Szegö type space equipped by the norm

$$||F||^2 = \sup_{0<x<1/2} \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(x+iy)|^2 dy. \tag{5.54}$$

Making use the Parseval equality for the Mellin transform (1.203) one can conclude that $F(z)$ is the Mellin transform of some function $\varphi(t) \in L_2(\mathbf{R}_+)$ such that

$$||F||^2 = \sup_{0<x<1/2} \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(x+iy)|^2 dy = \int_0^{\infty} |\varphi(t)|^2 dt. \tag{5.55}$$

Hence as is obvious the assumed property $F(z) = F(-z)$ implies that almost everywhere on $\mathbf{R}_+$ we have equality $\varphi(t) = \varphi(1/t)$. Therefore, returning now to the discussions in Section 2.4 imply equality (2.95) for the weighted function $\omega(x) = K_1(x)/2$. Hence invoking with (2.94) and (5.55) one can write the chain of equalities

$$||F||^2 = \int_0^{\infty} |\varphi(t)|^2 dt = \frac{1}{2} \int_0^{\infty} (f * \bar{f})(x) K_1(x) dx$$

$$= \frac{1}{2} \int_0^{\infty} \int_0^{\infty} K_1(u+v) f(u) \overline{f(v)} du dv = ||f||^2_{H_K}, \tag{5.56}$$

where the sign $*$ means the Laplace convolution (2.91) and $f(x)$ is some function from the convolution Hilbert space S, which coincides with H_K. Thus we arrive to the Paley-Wiener theorem for the Kontorovich-Lebedev transform (2.84).

Theorem 5.5. *An arbitrary function $F(z)$ of the variable $z = x+iy$ being satisfied the property $F(z) = F(-z)$ and analytic at the symmetric strip $|\Re z| < 1/2$ belongs to the Szegö type space* (5.54) *if and only if it has form* (2.84) *for some $f \in H_K$.*

Continue to estimate the L_p-norms of the Kontorovich-Lebedev transform (2.84) $K_z[f]$ and turn now to the norm (5.1). According to Lemma 2.5 under condition $f \in L_{\nu,p}(\mathbf{R}_+)$, $1 < p < 2, \nu < 1$ we find that $K_z[f]$ is analytic function at the strip $\{\mathcal{S}_\nu : |\Re z| < 1 - \nu\}$. Moreover, invoking with representation (2.120) one can extend composition (2.78) as follows

$$K_z[f] = \sqrt{\frac{\pi}{2}} \left[F\ e^{xu} [Lf](\cosh u)\right](y), \tag{5.57}$$

where $z = x + iy$, $|x| < 1 - \nu, y \in \mathbf{R}$ and the Fourier and the Laplace transforms are defined by formulae (1.191) and (1.215), respectively. Consequently, by virtue of inequality (1.193), where $q = p/(p-1)$ and the generalized Minkowski inequality (1.10) we have the estimate

$$\mu_q^{q/p} (K_{x+iy}[f], x) = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} |K_{x+iy}[f]|^q\, dy \right)^{1/p}$$

$$\leq \frac{(2\pi)^{(1-q)/2p}}{2^{1/p}} \left(\int_{-\infty}^{\infty} e^{pxu} \left| \int_0^{\infty} e^{-v\cosh u} f(v) dv \right|^p du \right)^{1/p}$$

$$\leq (2\pi)^{(1-q)/2p} \int_0^\infty K_{px}^{1/p}(pv)|f(v)|dv. \tag{5.58}$$

Continuing by using the Hölder inequality (1.8), we find

$$\mu_q^{q/p}\left(K_{x+iy}[f],x\right) \leq (2\pi)^{(1-q)/2p}||f||_{\nu,p}\left(\int_0^\infty K_{px}^{q/p}(pv)v^{q(1-\nu)-1}dv\right)^{1/q} < \infty \tag{5.59}$$

under condition $|x| < 1-\nu$. Thus we proved the following theorem.

Theorem 5.6. *Let $f \in L_{\nu,p}(\mathbf{R}_+)$, $1 < p < 2$, $\nu < 1$. Then for the Kontorovich-Lebedev transform $K_{x+iy}[f]$ the uniform estimate* (5.59) *of the H_q-norm like* (5.1) *at the strip $\{S_\nu : |x| < 1-\nu\}$ holds.*

Finally in this section we wish to show how to apply the results obtained above in Banach spaces of analytic functions for the Kontorovich-Lebedev transforms to some kind of the Mehler-Fock transform. As we already know from Chapter 3 this index transform connected with the K-L transform by means, for example its composition with the Hankel or the Laplace transforms. Clarify slightly about analytic properties of the Mehler-Fock transform kernel $P_{-1/2+i\tau}(z)$ as the function of a complex variable z , when τ is a fixed parameter. Indeed, by definition (1.55) we have the situation when the Gauss series (1.47) in this case in general diverges (compare with the definition of the Mehler-Fock transform (3.1), where the argument $2y^2+1$ of the Legendre function $P_{-1/2+i\tau}(2y^2+1)$ is always more than 1). Meanwhile, one can mean the respective Gauss hypergeometric function by the Mellin-Barnes type integral (1.46). Precisely, considering representation (1.84) for the function

$${}_2F_1\left(\frac{1}{2}+i\tau, \frac{1}{2}-i\tau; 1\ ; \frac{1-z}{2}\right) = P_{-1/2+i\tau}(z)$$

$$= \frac{\cosh(\pi\tau)}{2\pi^2 i}\int_{-i\infty}^{i\infty} \frac{\Gamma(s)\Gamma\left(\frac{1}{2}+i\tau-s\right)\Gamma\left(\frac{1}{2}-i\tau-s\right)}{\Gamma(1-s)}\left(\frac{z-1}{2}\right)^{-s}ds, \tag{5.60}$$

observe by virtue of the Slater Theorem 1.6 that the Mellin-Barnes integral (5.60) exists for all z with $|\arg(z\pm 1)| < \pi$. Moreover, this Legendre function is an analytic one at the complex plane $z = x+iy$ with the cut $-\infty < x < 1$. When $|1-z| < 2$ it has representation (5.60), while outside of this disc we have to use the corresponding formula (1.85) of the analytic continuation of the Gauss function.

Let us introduce now the following Mehler-Fock integral

$$\mathcal{MF}(z) = \pi\int_0^\infty \frac{\tau\tanh(\pi\tau)}{\cosh(\pi\tau)}P_{-1/2+i\tau}(z)f(\tau)d\tau. \tag{5.61}$$

Hence define the weighted Hilbert space $L_2(\mathbf{R}_+; \pi\tau\tanh(\pi\tau)/\cosh(\pi\tau))$ by the inner product

$$\langle f,g\rangle = \pi\int_0^\infty \frac{\tau\tanh(\pi\tau)}{\cosh(\pi\tau)}f(\tau)\overline{g(\tau)}d\tau. \tag{5.62}$$

In the similar manner one can calculate the reproducing kernel by formula from Lebedev [7] as the index integral from the product of two Legendre functions

$$P(z,\bar{u}) = \pi \int_0^\infty \frac{\tau \tanh(\pi\tau)}{\cosh(\pi\tau)} P_{-1/2+i\tau}(z) P_{-1/2+i\tau}(\bar{u}) d\tau = \frac{1}{z+\bar{u}}. \tag{5.63}$$

Thus we find the Szegö kernel (5.8) with $\nu = 1/2$ from the right-hand side of equality (5.63). The absolute convergence of index integral (5.63) for $\Re z, \Re u \geq 1$ undoubtedly follows from representation (3.2), asymptotic behavior of the Bessel function by formulae (1.92)-(1.93) and inequality (1.100).

Prove that the set of functions $\{P_{-1/2+i\tau}(z), \Re z \geq 1\}$ is complete in the Hilbert space (5.61). Indeed, integrating through by x in equality $\mathcal{MF}(x) \equiv 0$ for real $x \geq 1$ after multiplying it by e^{-px}, $p > 0$, change the order of integration by the Fubini theorem and calculate the inner integral by formula 2.17.7.1 from Prudnikov et al. [3]. As result we reduce it to the equality

$$\int_0^\infty \frac{\tau \tanh(\pi\tau)}{\cosh(\pi\tau)} K_{i\tau}(p) f(\tau) d\tau \equiv 0, \; p > 0. \tag{5.64}$$

Hence by previous discussions on the Kontorovich-Lebedev transform (5.21) we obtain the desired result that $f = 0$ almost everywhere on $\mathbf{R}_+$. Moreover, $f = 0$ a.e. if $\mathcal{MF}(z) \equiv 0$ for all $\Re z \geq 1$.

The following estimate being established by the Cauchy-Schwarz-Bunyakovskii inequality, namely we deduce

$$\begin{aligned} |\mathcal{MF}(z)| &\leq ||f||_{L_2(\mathbf{R}_+;\pi\tau\tanh(\pi\tau)/\cosh(\pi\tau))} \left(P(z,\bar{z})\right)^{1/2} \\ &= \frac{||f||_{L_2(\mathbf{R}_+;\pi\tau\tanh(\pi\tau)/\cosh(\pi\tau))}}{\sqrt{2x}}, \; z = x+iy. \end{aligned} \tag{5.65}$$

Furthermore, combining with Morera's theorem allow us to conclude that the function $F(z) = \mathcal{MF}(z)$ is analytic in the half-plane $\Re z \geq 1$. Also we constructed the reproducing kernel Hilbert space of the Szegö type, since as is evident

$$P(z,\bar{u}) = \frac{1}{z+\bar{u}} = \int_0^\infty e^{-zt} e^{-\bar{u}t} dt. \tag{5.66}$$

Hence according to Theorem 5.1 there exists some function $h \in L_2(\mathbf{R}_+)$ such that we have the identity

$$\frac{1}{2\pi} \int_{-\infty}^\infty |\mathcal{MF}(x+iy)|^2 dy = \int_0^\infty e^{-2xt} |h(t)|^2 dt, \; x \geq 1, \tag{5.67}$$

and the representation of type

$$\mathcal{MF}(z) = \int_0^\infty e^{-zt} h(t) dt = [Lh](z), \; \Re z \geq 1. \tag{5.68}$$

However, equality (5.68) admits an analytic continuation of the Mehler-Fock integral (5.61) at the right half-plane $\mathbf{R}^+$. Finally one can mean it as an analytic function at

$\mathbf{R}^+$ being satisfied the Paley-Wiener theorem 5.1. Thus, we established the following result.

Theorem 5.7. *The Mehler-Fock transform* (5.61) *of the function* $f \in L_2(\mathbf{R}_+; \pi\tau \tanh(\pi\tau)/\cosh(\pi\tau))$ *belongs to the Szegö space* (5.4) *if and only if it has form* (5.68) *for some* $h \in L_2(\mathbf{R}_+)$.

5.2 Composition theorems for general index transforms

This section deals with general index transforms of the Kontorovich-Lebedev type which have been slightly considered in Chapter 2. The key representation of this kind is contained in formula (2.127) where we gave the expression of the Kontorovich-Lebedev integral (2.1) in terms of the Mellin transform (1.203). First the generalization of the Kontorovich-Lebedev transform was demonstrated by Wimp [1], where the kernel function $\theta^*(s)$ in formula (2.128) of the general index kernel was taken in the form of the gamma-ratio (1.60) for Meijer's G-function (1.59). Thus the general integral transform by the index of the Meijer G-function arises. The respective inversion formula was simplified by the author in Yakubovich [2]. The corresponding expansion of an arbitrary function for the Wimp-Yakubovich transform being valued for the space of summable functions is exhibited by formula (1.237). This integral as is shown in Wimp [1], Yakubovich [4] gives a wide number as well as known and new examples of the index transforms with hypergeometric functions as the kernels. The class of index transforms of the Kontorovich-Lebedev type was selected and investigated by the author in Yakubovich [1]-[4], [13], Yakubovich and Luchko [2].

The key purpose of this section is to study the L_p-properties of these transforms being based on the L_p-theorems for the Mellin transform (1.203) in Chapter 1 and the results obtained above for the Kontorovich-Lebedev transform (2.1) in spaces $L_{\nu,p}(\mathbf{R}_+)$. As we know by Theorem 2.7 one can express the K-L transform of the $L_{\nu,p}$-function under conditions $1 < p \leq 2, \nu < 1$ through integral (2.127). Consequently, one can establish more complicated index transform by the functional replacement $f^*(s) \mapsto \theta^*(s)f^*(s)$ like within formula (2.129), where we introduced the index transform $Y^{\theta}_{i\tau}[f]$ of an arbitrary function f, meaning it as composition (2.133) under respective conditions for the function f and the kernel $\theta^*(s)$. The corresponding operator $[\Theta f]$ is a generalization of the Mellin convolution type integral transform (1.220) by the right-hand side of the Mellin-Parseval formula (1.214). Furthermore, this approach was developed in Samko et al. [1], Vu Kim Tuan et al. [1] for the so-called G-transform with the kernel $\theta^*(s)$ as the ratio of Euler's gamma-functions. As it occurs often this ratio increases power-exponentially at infinity like for example, for the inverse Laplace transform being obtained from representation (1.216) as well as for other Mellin convolution transforms. Nevertheless, the general index transform by formula (2.129) exists in view of the convergence of integral (2.128). Here we continue and develop results of Section 2.5 attracting our attention to the mapping and composition properties of general index transforms.

Let us return to composition (2.133). According to Theorem 2.8 it is true under conditions $f \in L_{\nu,p}(\mathbf{R}_+)$, $1 < p \leq 2$, $\nu < 1$ and $\theta^*(\nu + it) \in L_p\left(\mathbf{R}; e^{-\pi|t|/2}|t|^{-\nu}\right)$. However, we begin to study general transforms from reducing the integral representation (2.128) of the index kernel $Y^{\theta}_{i\tau}(x)$ to the index-convolution Kontorovich-Lebedev transform (2.150) of the function $\theta(x)$ being connected with $\theta^*(s)$ by means of the Mellin transform (1.203). Namely, the following result is fulfilled directly as the corollary of Theorem 1.17.

Theorem 5.8. *Let $\theta(x) \in L_{\nu,p}(\mathbf{R}_+)$, $1 < p \leq 2$, $\nu < 1$ be determined as preimage of the Mellin transform $\theta^*(s)$ by formula* (1.204), *where the convergence of the integral is meant by $L_{\nu,p}$-norm* (1.19). *Then the general index kernel $Y^{\theta}_{i\tau}(x)$, $x > 0$ defined as equality* (2.128) *is given by formula*

$$Y^{\theta}_{i\tau}(x) = KL[\theta](\tau, x) = \int_0^{\infty} K_{i\tau}(xy)\theta(y)dy. \tag{5.69}$$

Proof. With the Mellin-Barnes integral (2.124) for the Macdonald function $K_{i\tau}(x)$, where s runs through the contour $(1 - \nu - i\infty,\ 1 - \nu + i\infty)$, $\nu < 1$ and Stirling formula (1.33), it gives that for each $\tau \in \mathbf{R}_+$ the integrand of (2.124) as the gamma-product of two Euler's gamma-functions with the power multiplier, namely

$$2^{s-1}\Gamma\left(\frac{s + i\tau}{2}\right)\Gamma\left(\frac{s - i\tau}{2}\right),$$

belongs to the space $L_p(1 - \nu - i\infty, 1 - \nu + i\infty)$. Consequently, together with the condition $\theta(x) \in L_{\nu,p}(\mathbf{R}_+)$, $1 < p \leq 2$, $\nu < 1$ obtain that the Mellin-Parseval formula (1.214) can be immediately written for the kernel $Y^{\theta}_{i\tau}(x)$. Precisely, its left-hand side leads to the desired result (5.69). Theorem 5.8 is proved. •

As it is follows from (1.2), for $\nu \in \mathbf{R}$ we denote by $L_{\nu,\infty}(\mathbf{R}_+)$ the weighted space of Lebesgue measurable complex valued functions f for which

$$||f||_{\nu,\infty} = \operatorname*{ess\ sup}_{x \in \mathbf{R}_+} |x^{\nu} f(x)| < \infty. \tag{5.70}$$

One can estimate the $L_{\nu,p}$-norm of the kernel (5.69) by using inequality (1.100) and the generalized Minkowski inequality (1.10). In this case we obtain

Theorem 5.9. *Let $\theta(x)$ be from the space $L_{\nu,p}(\mathbf{R}_+)$, $\nu < 1$ and $p \geq 1$. Then for all $\tau \geq 0$ the following estimate of $L_{1-\nu,p}$-norm for the index kernel $Y^{\theta}_{i\tau}(x)$ is true*

$$||Y^{\theta}_{i\tau}||_{1-\nu,p} \leq 2^{-\nu-1}e^{-\delta\tau}\cos^{\nu-1}\delta\Gamma^2\left(\frac{1-\nu}{2}\right)||\theta||_{\nu,p}, \tag{5.71}$$

where $0 \leq \delta < \pi/2$.

Proof. Indeed, taking into account the simple interchange $xy = t$ write (5.69) in the form

$$Y_{i\tau}^{\theta}(x) = \frac{1}{x}\int_0^{\infty} K_{i\tau}(t)\theta\left(\frac{t}{x}\right) dt. \tag{5.72}$$

Hence, invoking with the generalized Minkowski inequality (1.10) and inequality (1.100) find that

$$||Y_{i\tau}^{\theta}||_{1-\nu,p} = \left(\int_0^{\infty} x^{-p\nu-1} dx \left|\int_0^{\infty} K_{i\tau}(t)\theta\left(\frac{t}{x}\right) dt\right|^p\right)^{1/p}$$

$$\leq e^{-\delta\tau}\int_0^{\infty} K_0(t\cos\delta)dt \left(\int_0^{\infty} x^{-p\nu-1}\left|\theta\left(\frac{t}{x}\right) dx\right|^p\right)^{1/p}, \ 0 \leq \delta < \pi/2. \tag{5.73}$$

Again change variable $y = t/x$. As a result it gives us that

$$||Y_{i\tau}^{\theta}||_{1-\nu,p} \leq e^{-\delta\tau}||\theta||_{\nu,p}\int_0^{\infty} t^{-\nu}K_0(t\cos\delta)dt$$

$$= 2^{-\nu-1}e^{-\delta\tau}\cos^{\nu-1}\delta\Gamma^2\left(\frac{1-\nu}{2}\right)||\theta||_{\nu,p}, \ \nu < 1, \tag{5.74}$$

by virtue of integral (2.125). Thus we obtained the desired result. This completes the proof of the Theorem 5.9. •

This theorem enables us to establish the mapping property of the general index transform introduced in (2.129). By applying the Hölder inequality (1.8) to the right-hand side of equality (2.129) we immediately find sufficient conditions of the boundedness of the operator $Y_{i\tau}^{\theta}[f]$ in the space $L_{\nu,p}(\mathbf{R}_+)$.

Theorem 5.10. *Let* $f(x) \in L_{\nu,p}(\mathbf{R}_+)$, $\nu < 1$, $p \geq 1$. *Let the function* $\theta(x)$ *be from the space* $L_{\nu,q}(\mathbf{R}_+)$, $q = p/(p-1)$. *Then the following uniform estimate for all* $\tau \geq 0, \delta \in [0, \pi/2)$ *holds*

$$\left|Y_{i\tau}^{\theta}[f]\right| \leq 2^{-\nu-1}e^{-\delta\tau}\cos^{\nu-1}\delta\Gamma^2\left(\frac{1-\nu}{2}\right)||\theta||_{\nu,q}||f||_{\nu,p}. \tag{5.75}$$

Proof. Invoking with definition (2.129) make use the Hölder inequality (1.8). Hence, in view of the previous theorem finally we obtain

$$\left|Y_{i\tau}^{\theta}[f]\right| \leq \int_0^{\infty}\left|Y_{i\tau}^{\theta}(t)f(t)\right| dt \leq ||Y_{i\tau}^{\theta}||_{1-\nu,q}||f||_{\nu,p}$$

$$\leq 2^{-\nu-1}e^{-\delta\tau}\cos^{\nu-1}\delta\Gamma^2\left(\frac{1-\nu}{2}\right)||\theta||_{\nu,q}||f||_{\nu,p}. \tag{5.76}$$

Theorem 5.10 is proved. •

Corollary 5.1. *The general index transform $Y^{\theta}_{i\tau}[f]$ is the bounded operator from the space $L_{\nu,p}(\mathbf{R}_+), \nu < 1,\ p \geq 1$ into the space $L_r(\mathbf{R}_+), r \geq 1$,*

$$||Y^{\theta}_{i\tau}[f]||_r \leq C||f||_{\nu,p}, \tag{5.77}$$

where

$$C = \frac{2^{-\nu-1}}{(\delta r)^{1/r}} \cos^{\nu-1}\delta\Gamma^2\left(\frac{1-\nu}{2}\right). \tag{5.78}$$

Proof. Indeed, this proposition it is not difficult to obtain from estimate (5.75), choosing and fixing some positive parameter $0 < \delta < \pi/2$. Hence integrating through in inequality (5.75) by $\tau \in \mathbf{R}_+$ in view of definition (1.1) of the norm it follows that

$$||Y^{\theta}_{i\tau}[f]||_r \leq 2^{-\nu-1}\cos^{\nu-1}\delta\Gamma^2\left(\frac{1-\nu}{2}\right)||f||_{\nu,p}$$

$$\times\left(\int_0^{\infty} e^{-\delta r\tau}d\tau\right)^{1/r}. \tag{5.79}$$

Thus the desired result comes after calculation of the integral in (5.79). Corollary 5.1 is proved. •

It is clear now to study the general index operator

$$Y^{\theta}_{i\tau}[f] = \int_0^{\infty} Y^{\theta}_{i\tau}(t)f(t)dt \tag{5.80}$$

we need to make connection of it with the Kontorovich-Lebedev transform (2.1) and use results of Chapter 2. First it is important the following composition theorem for operator (5.80).

Theorem 5.11. *Under conditions of Theorem 5.10 for all $\tau \geq 0$ the index transform (5.80) can be represented by formula*

$$Y^{\theta}_{i\tau}[f] = K_{i\tau}\left[(f * \theta)\right], \tag{5.81}$$

where the right-hand side of (5.81) means the composition of the Kontorovich-Lebedev transform (2.1) and the Mellin convolution (1.217) of functions f and θ.

Proof. Appealing to Theorem 1.20 we conclude that if $f \in L_{\nu,p}(\mathbf{R}_+)$, $\theta \in L_{\nu,q}(\mathbf{R}_+)$, $p^{-1}+q^{-1}=1$, then the operator

$$(f * \theta)(x) = \int_0^{\infty} \theta\left(\frac{x}{t}\right)f(t)\frac{dt}{t} \tag{5.82}$$

belongs to the space $L_{\nu,\infty}(\mathbf{R}_+)$. Therefore the Kontorovich-Lebedev transform within (5.81) exists. Hence substitute the expression of the index kernel $Y^{\theta}_{i\tau}(x)$ by formula

(5.69) in integral (5.80) changing beforehand the variable $xy = v$. According to estimate (5.76) one can perform to change the order of integration by Fubini's theorem. Thus we obtain

$$Y_{i\tau}^{\theta}[f] = \int_0^\infty f(t)\frac{dt}{t}\int_0^\infty K_{i\tau}(v)\theta\left(\frac{v}{t}\right)dv$$
$$= K_{i\tau}\left[(f * \theta)\right]. \tag{5.83}$$

Theorem 5.11 is proved. •

This theorem enables to mean the general index transform by the right-hand side of composition equality (5.81) and consequently, to use widely results related to the Kontorovich-Lebedev transform from Chapter 2. First of all let us appeal Theorem 2.2 of L_p-inversion of the K-L transform. If we assume that $0 < \nu < 1$, then the Mellin convolution (5.82) can be represented as follows

$$(f * \theta)(x) = \frac{2}{\pi^2}\text{l.i.m.}_{\varepsilon\to 0+}x^{\varepsilon-1}\int_0^\infty \tau\sinh((\pi-\varepsilon)\tau)K_{i\tau}(x)Y_{i\tau}^{\theta}[f]d\tau. \tag{5.84}$$

Hence let us take some function $\psi(x) \in L_{\nu,1}(\mathbf{R}_+)$. Then by definition (1.203) of the Mellin transform

$$\psi^*(s) = \int_0^\infty \psi(y)y^{s-1}dy,\ s = \nu + it, \tag{5.85}$$

we find that integral (5.85) is absolutely convergent and uniformly by $t \in \mathbf{R}$. Therefore, for all $s \in (\nu - i\infty, \nu + i\infty)$ we have the inequality $|\psi^*(s)| \leq C$, where C is an absolute positive constant. If we assume also the connection between $\psi^*(s)$ and $\theta^*(s)$ by means of the functional equation

$$\psi^*(s)\theta^*(s) = \frac{1}{1-s},\ s = \nu + it, \tag{5.86}$$

then one can establish the following result.

Theorem 5.12. *Let $f(x)$ be a function from the space $L_{\nu,p}(\mathbf{R}_+)$ with $0 < \nu < 1$ and $1 < p \leq 2$. Let $\theta(x) \in L_{\nu,q}(\mathbf{R}_+)$, $q = p/(p-1)$ and the function $\psi(x)$ be from the intersection of the spaces $L_{\nu,1}((0,a]) \cap L_1([a,\infty); \log x)$, where $a > 1$ is some fixed number. Then under assumption that the Mellin images of $\theta(x)$, $\psi(x)$ satisfy the functional equation (5.86) for each $x > 0$ the following inversion formula is valid*

$$\int_0^x f(y)dy = \frac{2}{\pi^2}\text{l.i.m.}_{\varepsilon\to 0+}\int_0^\infty \tau\sinh((\pi-\varepsilon)\tau)Y_{i\tau}^{\psi,\varepsilon}(x)Y_{i\tau}^{\theta}[f]d\tau, \tag{5.87}$$

where the kernel $Y_{i\tau}^{\psi,\varepsilon}(x)$ defined by formula

$$Y_{i\tau}^{\psi,\varepsilon}(x) = \int_0^\infty \psi\left(\frac{x}{v}\right)K_{i\tau}(v)v^{\varepsilon-2}dv. \tag{5.88}$$

Proof. First, let us explain our choice of the function $\psi(x)$ from the intersection $L_{\nu,1}((0,a]) \cap L_1([a,\infty); \log x)$, $a > 1$. In fact, as is obvious in this case $\psi(x) \in L_{\nu,1}(\mathbf{R}_+)$, $0 < \nu < 1$. Namely, we have

$$||\psi||_{\nu,1} = \int_0^\infty |\psi(y)| y^{\nu-1} dy$$

$$= \int_0^a |\psi(y)| y^{\nu-1} dy + \int_a^\infty |\psi(y)| y^{\nu-1} dy$$

$$\leq \int_0^a |\psi(y)| y^{\nu-1} dy + C \int_a^\infty |\psi(y)| \log y dy$$

$$= ||\psi||_{L_{\nu,1}((0,a])} + C ||\psi||_{L_1([a,\infty);\log x)} < \infty, \quad 0 < \nu < 1, \tag{5.89}$$

where C is a positive constant. Hence we need this condition to estimate the kernel (5.88). Indeed, invoking with inequality (1.100) we find that

$$\left|Y_{i\tau}^{\psi,\varepsilon}(x)\right| \leq e^{-\delta\tau} \int_0^\infty \left|\psi\left(\frac{x}{v}\right)\right| K_0(v\cos\delta) v^{\varepsilon-2} dv, \ \delta \in (0, \pi/2). \tag{5.90}$$

Further, by interchange $v^{-1} = u$ we have

$$\left|Y_{i\tau}^{\psi,\varepsilon}(x)\right| \leq e^{-\delta\tau} \int_0^\infty |\psi(xu)| K_0\left(\frac{\cos\delta}{u}\right) u^{-\varepsilon} du$$

$$\leq e^{-\delta\tau} \left(\int_0^a |\psi(xu)| K_0\left(\frac{\cos\delta}{u}\right) u^{-\varepsilon} du + \int_a^\infty |\psi(xu)| K_0\left(\frac{\cos\delta}{u}\right) u^{-\varepsilon} du\right)$$

$$= e^{-\delta\tau} \left(I_1(x,\varepsilon) + I_2(x,\varepsilon)\right). \tag{5.91}$$

Hence recall again the asymptotic behavior of the Macdonald function by formulae (1.96)-(1.97) and observe that uniformly by $\varepsilon \geq 0$

$$I_1(x,\varepsilon) = \int_0^a |\psi(xu)| K_0\left(\frac{\cos\delta}{u}\right) u^{-\varepsilon} du \leq C_1 ||\psi||_{L_{\nu,1}((0,a])}, \tag{5.92}$$

$$I_2(x,\varepsilon) = \int_a^\infty |\psi(xu)| K_0\left(\frac{\cos\delta}{u}\right) u^{-\varepsilon} du \leq C_2 ||\psi||_{L_1([a,\infty);\log x)}, \tag{5.93}$$

where C_i, $i = 1, 2$ are positive constants that depend only from fixed $x > 0$ and $\delta \in (0, \pi/2)$.

Now begin directly establish inversion formula (5.87). Since under conditions of this theorem $\psi(x) \in L_{\nu,1}(\mathbf{R}_+)$, then multiplying through in (5.84), namely in the equality

$$(f * \theta)(v) = \frac{2}{\pi^2} \text{l.i.m.}_{\varepsilon\to 0+} v^{\varepsilon-1} \int_0^\infty \tau \sinh((\pi-\varepsilon)\tau) K_{i\tau}(v) Y_{i\tau}^{\theta}[f] d\tau$$

by $\psi(x/v)/v$ and integrating through by $v \in \mathbf{R}_+$ we obtain

$$\int_0^\infty \psi\left(\frac{x}{v}\right) (f * \theta)(v) \frac{dv}{v}$$

$$= \frac{2}{\pi^2} \int_0^\infty \text{l.i.m.}_{\varepsilon \to 0+} \psi \left(\frac{x}{v}\right) v^{\varepsilon-1} \int_0^\infty \tau \sinh((\pi - \varepsilon)\tau) K_{i\tau}(v) Y^{\theta}_{i\tau}[f] d\tau dv. \tag{5.94}$$

Thus we organized in the left-hand side of (5.94) the iterated Mellin convolution (1.217). However, $(f * \theta)(x) \in L_{\nu,\infty}(\mathbf{R}_+)$. Therefore, as is known from Theorem 1.20 and properties of the Mellin convolution the operator $(\psi * (f * \theta))$ maps the space $L_{\nu,\infty}(\mathbf{R}_+)$ into itself and by Fubini's theorem it is easily to deduce the distributivity of the Mellin convolution, precisely the equality of kind

$$(\psi * (f * \theta))(x) = ((\psi * \theta) * f)(x), \; x > 0. \tag{5.95}$$

Furthermore, according to Theorem 1.17 it is not difficult to write an analog of the Mellin-Parseval equality (1.214) for the convolution (5.82) as

$$(f * \theta)(x) = \frac{1}{2\pi i} \int_{\nu - i\infty}^{\nu + i\infty} f^*(s) \theta^*(s) x^{-s} ds, \tag{5.96}$$

where the product $f^*(s)\theta^*(s) \in L_1(\nu - i\infty, \nu + i\infty)$ by virtue of the Hölder inequality (1.8). Consequently, substituting it into the left-hand side of (5.95), change the order of integration by the Fubini theorem in view of the absolute convergence of iterated integral $(\psi(x) \in L_{\nu,1}(\mathbf{R}_+))$. As result we obtain

$$(\psi * (f * \theta))(x) = \frac{1}{2\pi i} \int_{\nu - i\infty}^{\nu + i\infty} f^*(s) \theta^*(s) \psi^*(s) x^{-s} ds. \tag{5.97}$$

Functional equation (5.86) reduces it to

$$(\psi * (f * \theta))(x) = \frac{1}{2\pi i} \int_{\nu - i\infty}^{\nu + i\infty} \frac{f^*(s)}{1 - s} x^{-s} ds = \int_0^x f(y) dy. \tag{5.98}$$

Note, that the integral by y in (5.98) converges absolutely due to the estimate by using the Hölder inequality (1.8), namely

$$\int_0^x |f(y)| dy < ||f||_{\nu,p} \left(\int_0^x y^{(1-\nu)q-1} dy \right)^{1/q}$$

$$= \frac{x^{1-\nu}}{(q(1-\nu))^{1/q}} ||f||_{\nu,p}, x > 0, \; 0 < \nu < 1. \tag{5.99}$$

Let us treat now the right-hand side of (5.94). By boundedness of the operator within the integral by v, one can carry out the limit sign and change the order of integration by virtue of estimates (5.76), (5.92)-(5.93) and the Fubini theorem. Finally invoking with notation (5.88) we arrive to (5.87). This ends the proof of Theorem 5.12. •

5.3 Watson's type kernels

Here we consider slightly different index transforms, which involve compositions of the Kontorovich-Lebedev transform (2.1) and Mellin convolution transforms of the Watson type (1.220). One can occur such transforms apart from the previous ones in Section 5.1. Besides, we give sufficient conditions of their inversions drawing a parallel with the representation through the Kontorovich-Lebedev singular integral (2.42).

Let us introduce the following index transform

$$[S^{\varphi}_{i\tau}f] = \int_0^\infty S^{\varphi}_{i\tau}(t)f(t)dt, \quad \tau \geq 0, \tag{5.100}$$

where the kernel $S^{\varphi}_{i\tau}(x)$ is given by formula

$$S^{\varphi}_{i\tau}(x) = \int_0^\infty K_{i\tau}(y)\varphi(xy)dy, \quad x > 0. \tag{5.101}$$

In spite of the fact that index transform (5.100) can be reduced to the transform with kernel (5.69) by means of changes of variables and functions we attract our attention to these transforms separately. Furthermore, it gives us some advantages in obtaining suitable estimates below.

Theorem 5.13. *Let $\varphi(x)$ be from the space $L_{\nu,p}(\mathbf{R}_+)$, where $\nu < 1$ and $p \geq 1$. Then for all $\tau \geq 0$ the following estimate of $L_{\nu,p}$-norm for the index kernel $S^{\varphi}_{i\tau}(x)$ is true*

$$||S^{\varphi}_{i\tau}||_{\nu,p} \leq 2^{-\nu-1}\Gamma^2\left(\frac{1-\nu}{2}\right)\cos^{\nu-1}\delta e^{-\delta\tau}||\varphi||_{\nu,p}, \quad 0 \leq \delta < \pi/2. \tag{5.102}$$

Proof. By the same arguments as in previous section we obtain the chain of expressions

$$\begin{aligned}||S^{\varphi}_{i\tau}||_{\nu,p} &= \left(\int_0^\infty x^{p\nu-1}dx\left|\int_0^\infty K_{i\tau}(y)\varphi(xy)dy\right|^p\right)^{1/p}\\
&\leq \int_0^\infty |K_{i\tau}(y)|\,dy\left(\int_0^\infty x^{p\nu-1}\,|\varphi(xy)|^p\,dx\right)^{1/p}\\
&= ||\varphi||_{\nu,p}\int_0^\infty y^{-\nu}\,|K_{i\tau}(y)|\,dy\\
&\leq e^{-\delta\tau}||\varphi||_{\nu,p}\int_0^\infty K_0(y\cos\delta)y^{-\nu}dy\\
&= 2^{-\nu-1}\Gamma^2\left(\frac{1-\nu}{2}\right)\cos^{\nu-1}\delta e^{-\delta\tau}||\varphi||_{\nu,p}, \quad 0 \leq \delta < \pi/2.\end{aligned} \tag{5.103}$$

This completes the proof of Theorem 5.13. •

From Theorem 5.13 we immediately have the following result.

Theorem 5.14. *Let $\varphi \in L_{1-\nu,q}(\mathbf{R}_+)$ and $f(x)$ be from the space $L_{\nu,p}(\mathbf{R}_+), \nu > 0, p \geq 1, p^{-1} + q^{-1} = 1$. Then index transform* (5.100) *can be estimated by*

$$||[S_{i\tau}^{\varphi} f]|| \leq 2^{-\nu-2}\Gamma^2\left(\frac{\nu}{2}\right)\cos^{-\nu}\delta e^{-\delta\tau}||\varphi||_{1-\nu,q}||f||_{\nu,p}, \quad 0 \leq \delta < \pi/2. \tag{5.104}$$

We omit the proof of this theorem. Analogously Corollary 5.1 one can formulate

Corollary 5.2. *General index transform* (5.100) *is the bounded operator from the space $L_{\nu,p}(\mathbf{R}_+), \nu > 0, p \geq 1$ into the space $L_r(\mathbf{R}_+), r \geq 1$.*

Consider now the following operator

$$\left(I_\varepsilon^\psi g\right)(x) = \frac{2}{\pi^2}\int_0^\infty \tau \sinh((\pi - \varepsilon)\tau) S_{i\tau}^\psi(x) g(\tau) d\tau, \quad x > 0, \tag{5.105}$$

where $\varepsilon \in (0, \pi)$ and $S_{i\tau}^\psi(x)$ is index kernel (5.101) with a function $\psi(x)$.

Theorem 5.15. *Let $\psi(x) \in L_1((0,a]; \log x) \cap L_{\nu+1,1}([b, \infty)),\ a, b > 0$. Let φ be from $L_{1-\nu,q}(\mathbf{R}_+), \nu > 0, q \geq 1$. Then on the functions $g(\tau) = [S_{i\tau}^\varphi f]$ being represented by index transform* (5.100) *with the density $f(y) \in L_{\nu,p}(\mathbf{R}_+), p^{-1} + q^{-1} = 1$, operator* (5.105) *has the form*

$$\left(I_\varepsilon^\psi g\right)(x) = \frac{\sin\varepsilon}{\pi}\int_0^\infty\int_0^\infty \frac{uvK_1(\sqrt{u^2+v^2-2uv\cos\varepsilon})}{\sqrt{u^2+v^2-2uv\cos\varepsilon}}$$
$$\times \psi(xu)(\Phi f)(v) du dv, \tag{5.106}$$

where $(\Phi f)(x)$ is the operator of Watson's type (1.220)

$$(\Phi f)(x) = \int_0^\infty \varphi(xy) f(y) dy, \quad x > 0$$

with the kernel $\varphi(x)$.

Proof. First we show that for the index transform (5.100) under conditions of this theorem the following composition is true

$$[S_{i\tau}^\varphi f] = K_{i\tau}\left[(\Phi f)\right], \tag{5.107}$$

where $K_{i\tau}[f]$ is the Kontorovich-Lebedev transform (2.1) . In fact, it follows in the same manner as in previous section appealing to Theorem 5.14 and Fubini's theorem for the iterated integral in the right-hand side of (5.107). Further, substitute in (5.105) instead of $g(\tau)$ the expression of index transform (5.100) by formula (5.107) and instead of the kernel $S_{i\tau}^\psi(x)$ its definition like (5.101), respectively. Hence it becomes

$$\left(I_\varepsilon^\psi g\right)(x) = \frac{2}{\pi^2}\int_0^\infty \tau\sinh((\pi-\varepsilon)\tau)\int_0^\infty \psi(xv)K_{i\tau}(v)dv$$

$$\times \int_0^\infty K_{i\tau}(u) \int_0^\infty f(y)\varphi(uy)dydud\tau, \quad x > 0. \tag{5.108}$$

Make use inequality (1.100), the asymptotic behavior of the Macdonald function and the condition on the function $\psi(x)$ at the present theorem. As result we obtain for each $x > 0$ the estimate of the kernel $S_{i\tau}^{\psi}(x)$, namely

$$|S_{i\tau}^{\psi}(x)| \leq e^{-\delta\tau} \int_0^\infty |\psi(xv)| K_0(v\cos\delta) dv$$

$$\leq e^{-\delta\tau} \left[C_1 \int_0^1 |\psi(xv)| \log(v\cos\delta) dv + +C_2 \int_1^\infty |\psi(xv)| v^{\nu} dv \right]$$

$$\leq C_{x,\nu,\delta} e^{-\delta\tau}, \tag{5.109}$$

where $\delta \in (0, \pi/2)$ and all constants depend from x, ν, δ. From this invoke with (5.104) and consequently, for each $x > 0$ and $\varepsilon \in (0, \pi)$ integral (5.108) can be estimated as follows

$$\left| \left(I_{\varepsilon}^{\psi} g \right)(x) \right| \leq \frac{2^{-\nu-1}}{\pi^2} \Gamma^2 \left(\frac{\nu}{2} \right) \cos^{-\nu} \delta C_{x,\nu,\delta} ||\varphi||_{1-\nu,q} ||f||_{\nu,p}$$

$$\times \int_0^\infty \tau \sinh((\pi - \varepsilon)\tau) e^{-2\delta\tau} d\tau$$

$$= C_{x,\nu,\delta,\varepsilon} ||\varphi||_{1-\nu,q} ||f||_{\nu,p}, \tag{5.110}$$

where $\nu > 0, q^{-1} + p^{-1} = 1$, $C_{x,\nu,\delta,\varepsilon}$ is a positive constant and we may choose δ such that $(\pi - \varepsilon)/2 < \delta < \pi/2$. This gives the convergence of the integral by τ in (5.110). Therefore, perform to change the order of integration in (5.108) and calculate the integral by τ using formula (2.17). Thus we lead to representation (5.106). This ends the proof of Theorem 5.15. •

The inversion of the general index transform (5.100) in $L_{\nu,p}$ -spaces is given by

Theorem 5.16. *Let $1 < p \leq 2, 0 < \nu < 1$, $g(\tau) = [S_{i\tau}^{\varphi} f]$ for $f(x) \in L_{\nu,p}(\mathbf{R}_+) \cap L_{\nu,1}(\mathbf{R}_+)$ and $\varphi \in L_{1-\nu,q}(\mathbf{R}_+) \cap L_{1-\nu,1}(\mathbf{R}_+), q^{-1} + p^{-1} = 1$. Let a function $\psi(x)$ being satisfied the condition of the previous theorem belong to $L_{1+\nu,p}(\mathbf{R}_+)$. Then the limit equality of type*

$$\text{l.i.m.}_{\varepsilon \to 0+} \left(I_{\varepsilon}^{\psi} g \right)(x) = x^{-2} \int_0^x f(y) dy, \quad x > 0 \tag{5.111}$$

by the norm of $L_{1+\nu,p}$ is valid if and only if

$$\psi^*(1+s)\varphi^*(1-s) = (1-s)^{-1}, \quad \Re s = \nu, \tag{5.112}$$

where the sign " $*$ " denotes the Mellin transform (1.203) of functions $\psi(x)$ and $\varphi(x)$. Furthermore, the limit in (5.111) exists also almost everywhere on $\mathbf{R}_+$.

Proof. First consider integral (5.106). After changing of variables $v = u(\cos\varepsilon + t\sin\varepsilon), u = u$ we immediately obtain

$$\left(I_{\varepsilon}^{\psi} g \right)(x) = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty \frac{R(t,u,\varepsilon)}{t^2+1} u\psi(xu(\cos\varepsilon + t\sin\varepsilon))$$

$$\times(\cos\varepsilon + t\sin\varepsilon)(\Phi f)(u)du dt, \quad \varepsilon \in (0,\pi), \tag{5.113}$$

where

$$R(t,u,\varepsilon) = \begin{cases} u\sin\varepsilon(t^2+1)^{1/2}K_1(u\sin\varepsilon(t^2+1)^{1/2}), \ t \geq -\cot\varepsilon, \\ 0, \ t < -\cot\varepsilon. \end{cases} \tag{5.114}$$

By the same arguments as in (2.21) for any $t \in \mathbf{R}$, $u \in \mathbf{R}_+$, and $\varepsilon \in (0,\pi)$, we have $|R(t,u,\varepsilon)| < C$, where C is a positive constant. Moreover,

$$\lim_{\varepsilon\to 0+} R(t,u,\varepsilon) = 1.$$

Hence one can estimate the norm of operator (5.113) in the space $L_{\nu,p}(\mathbf{R}_+)$ for each fixed $\varepsilon \in (0,\pi)$ by using the generalized Minkowski inequality (1.10) and conditions of this theorem. We have

$$\left|\left|\left(I_\varepsilon^\psi g\right)\right|\right|_{\nu+1,p} \leq \frac{C}{\pi}\int_0^\infty u|(\Phi f)(u)|$$

$$\times\int_{-\cot\varepsilon}^\infty \frac{||\psi(xu(\cos\varepsilon + t\sin\varepsilon))||_{\nu+1,p}}{t^2+1}(\cos\varepsilon + t\sin\varepsilon)dt du$$

$$\leq \frac{C}{\pi}||\psi||_{\nu+1,p}\int_0^\infty u^{-\nu}\int_0^\infty |f(y)\varphi(uy)|dy du$$

$$\times\int_{-\cot\varepsilon}^\infty \frac{(\cos\varepsilon + t\sin\varepsilon)^{-\nu}}{t^2+1}dt \leq C_1||\psi||_{\nu+1,p}||\varphi||_{1-\nu,1}||f||_{\nu,1}, \tag{5.115}$$

where C and C_1 are positive constants. The last integral by t is convergent when $0 < \nu < 1$ by virtue of formula 3.252.12 in Gradshtein and Ryzhik [1] (see also (2.181)), precisely

$$I_\varepsilon = \int_{-\cot\varepsilon}^\infty \frac{(\cos\varepsilon + t\sin\varepsilon)^{-\nu}}{t^2+1}dt$$

$$= \sin^{-\nu}\varepsilon\int_0^\infty \frac{v^{-\nu}}{(v-\cot\varepsilon)^2+1}dv = \frac{\pi\sin(\nu\varepsilon)}{\sin(\pi\nu)}. \tag{5.116}$$

Further, since $f(x) \in L_{\nu,p}(\mathbf{R}_+), p > 1$, then one can show that

$$x^{-2}\int_0^x f(y)dy \in L_{\nu+1,p}(\mathbf{R}_+). \tag{5.117}$$

Indeed, it follows straightforward from the generalized Minkowski inequality

$$\left|\left|x^{-2}\int_0^x f(y)dy\right|\right|_{\nu+1,p} = \left|\left|x^{-1}\int_0^1 f(xy)dy\right|\right|_{\nu+1,p}$$

$$\leq \int_0^1 dy\left(\int_0^\infty |x^\nu f(xy)|^p\frac{dx}{x}\right)^{1/p} = ||f||_{\nu,p}\int_0^1 y^{-\nu}dy < \infty, \ 0 < \nu < 1. \tag{5.118}$$

Consequently, appealing again to the properties of the Poisson kernel (1.14) from representation (5.113) we obtain that

$$\left|\left|\left(I_\varepsilon^\psi g\right) - x^{-2}\int_0^x f(y)dy\right|\right|_{\nu+1,p} \leq \frac{1}{\pi}\int_{-\infty}^\infty \frac{|\cos\varepsilon + t\sin\varepsilon|}{t^2+1}dt\left|\left|\int_0^\infty u(\Phi f)(u)\right.\right.$$

$$\times R(t,u,\varepsilon)\psi(xu(\cos\varepsilon + t\sin\varepsilon))du - x^{-2}\int_0^x f(y)dy\Big\|_{\nu+1,p} dt. \tag{5.119}$$

It is clear our desire now to establish that the right-hand side of inequality (5.119) tends to zero, when $\varepsilon \to 0+$, $1 < p \le 2$. This fact evidently means limit equality (5.111). Indeed, in view of the above estimates we perform to use the Lebesgue dominated convergence Theorem 1.2 and the property of the continuity of $L_{\nu,p}$-norm. Hence we find that the right-hand side of (5.119) tends to the following expression

$$\left\|\int_0^\infty u(\Phi f)(u)\psi(xu)du - x^{-2}\int_0^x f(y)dy\right\|_{\nu+1,p}. \tag{5.120}$$

Our purpose now to show that it is equal to zero, when $1 < p \le 2$. As it follows straightforward due to conditions of this theorem and from the Mellin-Parseval equality (1.214) the Watson type operator (Φf) can be written as

$$(\Phi f)(x) = \frac{1}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty} f^*(s)\varphi^*(1-s)x^{s-1}ds, \tag{5.121}$$

where $\varphi^*(s)$ is the Mellin transform (1.203) of the function $\varphi(x)$ and $\varphi^*(1-\nu-it) \in L_p(\mathbf{R}), 1 < p \le 2$. Since $f^*(\nu+it) \in L_q(\mathbf{R}), p^{-1}+q^{-1} = 1$, then by the Hölder inequality (1.8) we conclude that integral (5.121) is absolutely convergent. Moreover, the following inequality immediately can be deduced from (5.121)

$$|(\Phi f)(x)| < Cx^{\nu-1}, x > 0. \tag{5.122}$$

Meanwhile clearly, that our assumption $\psi(x) \in L_1((0,a];\log x)\cap L_{\nu+1,1}([b,\infty))$, $a, b > 0$ provides the condition $\psi(x) \in L_{1+\nu,1}(\mathbf{R}_+)$. It allows us to change the order of integration, substituting (5.121) into the integral in (5.120). Thus invoking with (5.112), we obtain the chain of equalities

$$\left\|\int_0^\infty u(\Phi f)(u)\psi(xu)du - x^{-2}\int_0^x f(y)dy\right\|_{\nu+1,p}$$

$$= \left\|\frac{1}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty} f^*(s)\psi^*(1+s)\varphi^*(1-s)x^{-s-1}ds - x^{-2}\int_0^x f(y)dy\right\|_{\nu+1,p}$$

$$= \left\|\frac{1}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty} \frac{f^*(s)}{1-s}x^{-s-1}ds - x^{-2}\int_0^x f(y)dy\right\|_{\nu+1,p} = 0. \tag{5.123}$$

Note, that we used representation (1.213) to establish the last equality in (5.123). Conversely, if

$$\left\|\frac{1}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty} f^*(s)\psi^*(1+s)\varphi^*(1-s)x^{-s-1}ds - x^{-2}\int_0^x f(y)dy\right\|_{\nu+1,p} = 0, \tag{5.124}$$

then almost for all $x > 0$

$$\int_0^x f(y)dy = \frac{1}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty} f^*(s)\psi^*(1+s)\varphi^*(1-s)x^{1-s}ds. \tag{5.125}$$

However, Theorem 1.16 show that it is possible only if almost everywhere on $(\nu - i\infty, \nu + i\infty)$ the relation (5.112) is to be satisfied. Thus it leads to the desired limit equality (5.111). The existence of the limit almost everywhere on $\mathbf{R}_+$ follows from the radial property of the Poisson kernel (1.14). Theorem 5.16 is proved. •

5.4 Compositions with the Mellin-Barnes integrals

Throughout this section we deal with the index transforms being represented by composition (5.107). However, the operator (Φf) is the Mellin-Barnes integral of type

$$(\Phi f)(x) = \frac{1}{2\pi i}\int_{1-\nu-i\infty}^{1-\nu+i\infty} \varphi^*(s) f^*(1-s) x^{-s} ds, \quad x > 0. \tag{5.126}$$

As is known the Mellin-Parseval formula (1.214) gives us such representation provided that Mellin reimages exist, for instance, by Theorem 1.16 and belong to the corresponding L_p-spaces. Nevertheless, in the case of index transforms of the Kontorovich-Lebedev type we have to take into account that integral (1.213) for the function $\varphi^*(s)$ can be divergent in spite of the fact, that the respective integral (2.128) as for the function $\theta(s)$ remains even absolutely convergent (see Theorem 2.8). For example, if $\varphi^*(s) = [\Gamma(s/2)]^{-1}$, then $\varphi^*(s)$ does not belong any space $L_p(\nu - i\infty, \nu + i\infty)$. However, one can substitute it in formula (2.128) instead of $\theta^*(s)$ and observe its absolute convergence, which can be easily verified by Stirling's formula (1.32). In particular, by the similar way the general G-transform of the Mellin convolution type was introduced (see details in Samko et al. [1], Vu Kim Tuan et al. [1]). Note, that this approach is also worth mentioning in our considerations below.

So we begin setting the following result.

Theorem 5.17. *Let $f(x)$ be from the space $L_{\nu,p}(\mathbf{R}_+)$, where $\nu > 0, 1 < p \le 2$ and let $\varphi^*(1-\nu+it)e^{-\pi|t|/2}|t|^{\nu-1} \in L_p(-\infty,\infty)$. If $\varphi^*(s)f^*(1-s) \in L_p(1-\nu-i\infty, 1-\nu+i\infty)$, then composition* (5.107) *with operator* (5.126) *can be represented by formula*

$$[S_{i\tau}^{\varphi} f] = \frac{1}{4\pi i}\int_{1-\nu-i\infty}^{1-\nu+i\infty} 2^{-s}\Gamma\left(\frac{1-s+i\tau}{2}\right)\Gamma\left(\frac{1-s-i\tau}{2}\right)\varphi^*(s) f^*(1-s) ds. \tag{5.127}$$

In addition, this composition has form (5.100) *with the kernel*

$$S_{i\tau}^{\varphi}(x) = \frac{1}{4\pi i}\int_{1-\nu-i\infty}^{1-\nu+i\infty} \Gamma\left(\frac{1-s+i\tau}{2}\right)\Gamma\left(\frac{1-s-i\tau}{2}\right)\varphi^*(s)(2x)^{-s} ds, \quad x > 0. \tag{5.128}$$

Proof. To deduce (5.127) apply the Mellin-Parseval equality (1.214). Indeed, due to the condition $\varphi^*(s)f^*(1-s) \in L_p(1-\nu-i\infty, 1-\nu+i\infty)$ and Theorem 1.16 it follows that $(\Phi f)(x) \in L_{1-\nu,q}(\mathbf{R}_+)$ with $q = p/(p-1)$ and the convergence of integral (5.126) is meant by the $L_{1-\nu,q}$-norm. Meanwhile, by using directly the asymptotic behavior of the Macdonald function by formulae (1.96)-(1.97) observe that for each $\tau \in \mathbf{R}_+$ we have $K_{i\tau}(x) \in L_{\nu,p}(\mathbf{R}_+), \nu > 0, 1 < p \leq 2$. Consequently, the desired formula (5.127) immediately arises, invoking with the Mellin-Barnes integral (2.124).

Furthermore, according to condition $\varphi^*(1-\nu+it)e^{-\pi|t|/2}|t|^{\nu-1} \in L_p(-\infty,\infty)$ and Stirling's formula (1.33) conclude that the product $\varphi^*(1-\nu+it)e^{-\pi|t|/2}|t|^{\nu-1}$ asymptotically coincides with the integrand of (5.128). Therefore, appealing again to the Mellin-Parseval formula (1.214) and taking into account the condition $f \in L_{\nu,p}(\mathbf{R}_+)$, $\nu > 0, 1 < p \leq 2$ arrive to (5.100) with the kernel (5.128). This completes the proof of Theorem 5.17. •

Considering the boundedness of the operator of index transform (5.127) in the space $L_r(\mathbf{R}_+)$, $r \geq 1$ we need to establish an analog of inequality (5.104). However, in this case it can be reduced to (5.104), when we assume additionally that $\varphi^*(1-\nu+it) \in L_q(\mathbf{R}_+)$. Nevertheless, one can estimate the norm of operator (5.127) in term of the Watson type operator (5.126). Namely, making use composition representation (5.107) we arrive to the following theorem.

Theorem 5.18. *The index transform, given by formula* (5.127) *is a bounded operator in the space* $L_r(\mathbf{R}_+), r \geq 1$. *Moreover, the following estimate takes place*

$$\|\,[S^{\varphi}_{i\tau}f]\,\|_{L_r} \leq C\|(\Phi f)\|_{1-\nu,q} < \infty, \quad q > 1, \tag{5.129}$$

where C is a some constant.

Proof. First to show inequality (5.129) it is sufficient to take the composition (5.107) and to apply the Hölder inequality (1.8). Thus as above we obtain

$$\|[S^{\varphi}_{i\tau}f]\| \leq \int_0^\infty |K_{i\tau}(t)(\Phi f)(t)|\,dt \leq e^{-\tau\delta}\int_0^\infty K_0(t\cos\delta)\,|(\Phi f)(t)|\,dt$$

$$\leq e^{-\delta\tau}\left(\int_0^\infty t^{\nu p-1}K_0^p(t\cos\delta)dt\right)^{1/p}\|(\Phi f)\|_{1-\nu,q} < \infty, \tag{5.130}$$

provided that $0 < \delta < \pi/2, \nu > 0$. The fact $(\Phi f) \in L_{1-\nu,q}(\mathbf{R}_+)$ follows from previous Theorem 5.17. Consequently, integrating through by $\tau \in \mathbf{R}_+$ in (5.130) in view of (1.1) we have the relation

$$\|\,[S^{\varphi}_{i\tau}f]\,\|_{L_r} = \left(\int_0^\infty \|[S^{\varphi}_{i\tau}f]\|^r\,d\tau\right)^{1/r}$$

$$\leq C_{\nu,p}\|(\Phi f)\|_{1-\nu,q}\left(\int_0^\infty e^{-\delta r\tau}d\tau\right)^{1/r} = C\|(\Phi f)\|_{1-\nu,q} \tag{5.131}$$

which leads us to (5.129). Theorem 5.18 is proved. •

Now one can formulate corresponding Theorems 5.15-5.16 in the case of operator (5.126), slightly changing conditions accordingly to the Theorems 5.17-5.18.

Theorem 5.19. *Let ψ satisfy the assumption of Theorem* 5.15. *Then under conditions of Theorem* 5.17 *on the functions $g(\tau) = [S_{i\tau}^{\varphi} f]$ representation* (5.106) *of the approximation operator* (5.105) *is fulfilled.*

Remark 5.1. The proof of this theorem and detailed verification we leave for the reader. Note only, that it is a slight change in the proof of Theorem 5.15 and we may start from estimation of the following iterated integral

$$\left(I_\varepsilon^\psi g\right)(x) = \frac{2}{\pi^2}\int_0^\infty \tau \sinh((\pi-\varepsilon)\tau)\int_0^\infty \psi(xv)K_{i\tau}(v)dv$$

$$\times \int_0^\infty K_{i\tau}(u)(\Phi f)(u)dud\tau, \quad x>0. \tag{5.132}$$

Theorem 5.20. *Let $1 < p \le 2, 0 < \nu < 1, g(\tau) = [S_{i\tau}^{\varphi} f]$ be under conditions of Theorem* 5.17 *for $f(x) \in L_{\nu,p}(\mathbf{R}_+)$. Then for $\psi(x) \in L_1((0,a];\log x)\cap L_{\nu+1,1}([b,\infty))\cap L_{1+\nu,p}(\mathbf{R}_+)$, $a,b>0$, and $(\Phi f)(x) \in L_{1-\nu,1}(\mathbf{R}_+)$ equality* (5.111) *is true if and only if functions ψ^*, φ^* satisfy equation* (5.112) *almost for all $s \in (\nu - i\infty, \nu + i\infty)$. Besides the limit in* (5.111) *exists almost everywhere on $\mathbf{R}_+$.*

Remark 5.2. The proof of Theorem 5.20 can be established in the same manner as in Theorem 5.16. However, we need here to assume that $(\Phi f)(x) \in L_{1-\nu,1}(\mathbf{R}_+)$. In this case, for example estimate (5.115) takes form

$$\left|\left|\left(I_\varepsilon^\psi g\right)\right|\right|_{\nu+1,p} \le \frac{C}{\pi}\int_0^\infty u|(\Phi f)(u)|$$

$$\times \int_{-\cot\varepsilon}^\infty \frac{||\psi(xu(\cos\varepsilon + t\sin\varepsilon))||_{\nu+1,p}}{t^2+1}(\cos\varepsilon + t\sin\varepsilon)dtdu$$

$$\le \frac{C}{\pi}||\psi||_{\nu+1,p}\int_0^\infty u^{-\nu}|(\Phi f)(u)|du$$

$$\times \int_{-\cot\varepsilon}^\infty \frac{(\cos\varepsilon + t\sin\varepsilon)^{-\nu}}{t^2+1}dt \le C_1||\psi||_{\nu+1,p}||(\Phi f)||_{1-\nu,1}. \tag{5.133}$$

Finally note , that in subsequent chapters we shall exhibit some examples of general index transforms which contain hypergeometric functions as the kernels.

Chapter 6

Index Transforms of The Lebedev-Skalskaya Type

In this chapter we examine separately enough wide class of the index transforms generated by the following operators

$$[\Re f](\tau) = \int_0^\infty \Re K_z(t) f(t) dt, \ z = \alpha + i\tau, \tag{6.1}$$

$$[\Im f](\tau) = \int_0^\infty \Im K_z(t) f(t) dt, \ z = \alpha + i\tau, \tag{6.2}$$

where we mean as usually by

$$\Re K_z(t) = \frac{K_{\alpha+i\tau}(t) + K_{\alpha-i\tau}(t)}{2}, \tag{6.3}$$

$$\Im K_z(t) = \frac{K_{\alpha+i\tau}(t) - K_{\alpha-i\tau}(t)}{2i}, \tag{6.4}$$

the real and imaginary parts respectively, of the Macdonald function (1.91) of the complex index $z = \alpha + i\tau, \tau \in \mathbf{R}$ with fixed real parameter α and variable $t \in \mathbf{R}_+$. One can draw a parallel with the Fourier transform (1.191) and the corresponding cosine and sine Fourier transforms (1.197)-(1.198). Here we have in general the Kontorovich-Lebedev transform (2.84) and investigate its $\Re$- and $\Im$-analogs as the index transforms (6.1)-(6.2).

First these transforms appear in Lebedev and Skalskaya [3], namely it was considered the case $z = 1/2 + i\tau$ in connection with applications to the related problems in mathematical physics. There it was shown the composition structure of the Lebedev-Skalskaya transforms by means of their representations through the Kontorovich-Lebedev transform (2.1) and the Riemann-Liouville fractional integral of the order $\alpha = 1/2$. We remark here that miscellaneous results on the fractional integro-differential operators are contained in the monograph of Samko et al. [1]. Later the Lebedev-Skalskaya transforms were investigated by Rappoport [1], Poruchikov and Rappoport [1]. Detail consideration of these transforms in L_p has been given recently by the author in Yakubovich and Luchko [2]. The attempt to spread the

Wimp approach on the Lebedev-Skalskaya transforms is undertaken in Yakubovich et al. [1, 1994].

6.1 Useful representations and estimates

The aim of this section is to provide necessary integral representations and miscellaneous estimates for kernels (6.3)-(6.4) and apply it to study below the mapping, inversion and convolution properties of the Lebedev-Skalskaya type index transforms.

We begin from integrals (1.98), (2.83) and (4.47) related to the Macdonald function (1.91) to obtain the respective representations for functions (6.3)-(6.4). Indeed, taking, for instance integral (4.47) and using elementary trigonometric formulae we immediately deduce that

$$\Re K_{\alpha+i\tau}(x) = \int_0^\infty e^{-x\cosh v} \cosh(\alpha v)\cos(\tau v)dv, \tag{6.5}$$

$$\Im K_{\alpha+i\tau}(x) = \int_0^\infty e^{-x\cosh v} \sinh(\alpha v)\sin(\tau v)dv, \tag{6.6}$$

where we mean $x > 0$ and $\alpha \in \mathbf{R}$. Hence it is not difficult to obtain the following uniform by $\tau \in \mathbf{R}$ inequality

$$\left| \begin{Bmatrix} \Re \\ \Im \end{Bmatrix} K_{\alpha+i\tau}(x) \right| \leq \int_0^\infty e^{-x\cosh v} \cosh(\alpha v)dv = K_\alpha(x). \tag{6.7}$$

We need to comprise also the case of inequality (1.100) for the Macdonald function. Consequently, in the same manner by analytic continuation of the integrand in (6.5)-(6.6) on the strip $\beta \in (i\delta - \infty, i\delta + \infty)$ with $\delta \in [0, \pi/2)$ one can write key formulae

$$\Re K_{\alpha+i\tau}(x) = \frac{1}{2}\int_{i\delta-\infty}^{i\delta+\infty} e^{-x\cosh\beta + i\tau\beta}\cosh(\alpha\beta)d\beta, \ x > 0, \tag{6.8}$$

$$\Im K_{\alpha+i\tau}(x) = \frac{1}{2}\int_{i\delta-\infty}^{i\delta+\infty} e^{-x\cosh\beta + i\tau\beta}\sinh(\alpha\beta)d\beta, \ x > 0. \tag{6.9}$$

Therefore we find the inequalities being involved as follows

$$\left| \begin{Bmatrix} \Re \\ \Im \end{Bmatrix} K_{\alpha+i\tau}(x) \right| \leq \frac{e^{-\delta\tau}}{2}\int_{-\infty}^\infty e^{-x\cos\delta\cosh v} \cosh(\alpha v)dv = e^{-\delta\tau}K_\alpha(x\cos\delta), \tag{6.10}$$

where $\delta \in [0, \pi/2)$, $x > 0$. In particular, when $\alpha = 1/2$ it becomes

$$\left| \begin{Bmatrix} \Re \\ \Im \end{Bmatrix} K_{1/2+i\tau}(x) \right| \leq \sqrt{\frac{\pi}{2x\cos\delta}} e^{-\delta\tau - x\cos\delta}. \tag{6.11}$$

Considering integral representation (2.83) immediately obtain the formulae

$$\Re K_{\alpha+i\tau}(x)=\frac{1}{2}\int_0^\infty e^{-x(v+v^{-1})/2}v^{\alpha-1}\cos(\tau\log v)dv, \tag{6.12}$$

$$\Im K_{\alpha+i\tau}(x)=\frac{1}{2}\int_0^\infty e^{-x(v+v^{-1})/2}v^{\alpha-1}\sin(\tau\log v)dv, \tag{6.13}$$

that can be deduced directly from the above representations by the interchange $v = e^u$.

Let us touch now some reasonable formulae for the product of the functions $\Re K_{\alpha+i\tau}(x)$, $\Im K_{\alpha+i\tau}(x)$ of different arguments. Our start point is the Macdonald formula (1.103). So it is clear that one can write the identities as

$$\Re\left[K_{\alpha+i\tau}(x)K_{\alpha+i\tau}(y)\right]=\frac{1}{2}\int_0^\infty \exp\left(-\frac{1}{2}\left(\frac{xy}{u}+\frac{xu}{y}+\frac{uy}{x}\right)\right)\Re K_{\alpha+i\tau}(u)\frac{du}{u}, \tag{6.14}$$

$$\Im\left[K_{\alpha+i\tau}(x)K_{\alpha+i\tau}(y)\right]=\frac{1}{2}\int_0^\infty \exp\left(-\frac{1}{2}\left(\frac{xy}{u}+\frac{xu}{y}+\frac{uy}{x}\right)\right)\Im K_{\alpha+i\tau}(u)\frac{du}{u}. \tag{6.15}$$

Turn first to formula (6.14). The left-hand side of it yields

$$\Re\left[K_{\alpha+i\tau}(x)K_{\alpha+i\tau}(y)\right]=\Re K_{\alpha+i\tau}(x)\Re K_{\alpha+i\tau}(y)$$
$$-\Im K_{\alpha+i\tau}(x)\Im K_{\alpha+i\tau}(y). \tag{6.16}$$

Further, one can calculate the following useful integral that slightly different from the right-hand side of (6.14), namely

$$I_\alpha(x,y,\tau)=\frac{1}{2}\int_0^\infty \exp\left(-\frac{1}{2}\left(\frac{xy}{u}+\frac{xu}{y}+\frac{uy}{x}\right)\right)\Re K_{\alpha-1+i\tau}(u)du. \tag{6.17}$$

For this begin from formula (4.82), meaning instead of the index α the complex index $\alpha+i\tau$. Hence observe, that it is possible the differentiation through in (4.82) by the parameter β in view of the uniform convergence of the integral by $\beta \geq 1$. Therefore, making use the identity in Erdélyi et al. [1]

$$x\frac{d}{dx}K_\nu(x)=-K_\nu(x)-xK_{\nu-1}(x), \tag{6.18}$$

setting $\beta = 1$ and invoking with the Macdonald formula (1.103) find that the left-hand side of (4.82) generates the equality

$$\frac{1}{2}\int_0^\infty \exp\left(-\frac{1}{2}\left(\frac{xy}{u}+\frac{xu}{y}+\frac{uy}{x}\right)\right)\frac{d}{du}K_{\alpha+i\tau}(u)du$$
$$=-K_{\alpha+i\tau}(x)K_{\alpha+i\tau}(y)$$
$$-\frac{1}{2}\int_0^\infty \exp\left(-\frac{1}{2}\left(\frac{xy}{u}+\frac{xu}{y}+\frac{uy}{x}\right)\right)K_{\alpha-1+i\tau}(u)du. \tag{6.19}$$

Meanwhile, differentiating the right-hand side of (4.82) by β and letting there $\beta = 1$ it is not difficult to deduce from (6.18)-(6.19) the final expression as

$$\frac{1}{2}\int_0^\infty \exp\left(-\frac{1}{2}\left(\frac{xy}{u}+\frac{xu}{y}+\frac{uy}{x}\right)\right) K_{\alpha-1+i\tau}(u)du$$

$$= \frac{xy}{x^2-y^2}\left[xK_{\alpha-1+i\tau}(y)K_{\alpha+i\tau}(x) - yK_{\alpha-1+i\tau}(x)K_{\alpha+i\tau}(y)\right]. \qquad (6.20)$$

Consequently, taking the real part through in equality (6.20) we obtain the value of integral (6.17), namely

$$I_\alpha(x,y,\tau) = \frac{xy}{x^2-y^2}\left[x\Re\left[K_{\alpha-1+i\tau}(y)K_{\alpha+i\tau}(x)\right] - y\Re\left[K_{\alpha-1+i\tau}(x)K_{\alpha+i\tau}(y)\right]\right]. \quad (6.21)$$

In the case $\alpha = 1/2$ one can reduce formula (6.21) to more simple form. Indeed, invoke with the evenness of the Macdonald function by its index and find that $K_{-1/2+i\tau}(x) = K_{1/2-i\tau}(x) = K_{\bar{z}}(x)$, where $z = 1/2 + i\tau$. Therefore, equality (6.21) can be simplified to the identity

$$I_{1/2}(x,y,\tau) = \frac{xy}{x+y}\left[\Re K_{1/2+i\tau}(x)\Re K_{1/2+i\tau}(y) + \Im K_{1/2+i\tau}(x)\Im K_{1/2+i\tau}(y)\right]. \quad (6.22)$$

Hence, combining with (6.14) obtain the analog of the Macdonald formula (1.103) for $\Re$-functions

$$\Re K_{1/2+i\tau}(x)\Re K_{1/2+i\tau}(y)$$

$$= \frac{1}{4}\int_0^\infty \exp\left(-\frac{1}{2}\left(\frac{xy}{u}+\frac{xu}{y}+\frac{uy}{x}\right)\right)\Re K_{1/2+i\tau}(u)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{u}\right)du. \qquad (6.23)$$

Similarly, appealing to (6.16) the related formula with the product of $\Im$-functions in the left-hand side is straightforwarded, namely

$$\Im K_{1/2+i\tau}(x)\Im K_{1/2+i\tau}(y)$$

$$= \frac{1}{4}\int_0^\infty \exp\left(-\frac{1}{2}\left(\frac{xy}{u}+\frac{xu}{y}+\frac{uy}{x}\right)\right)\Re K_{1/2+i\tau}(u)\left(\frac{1}{x}+\frac{1}{y}-\frac{1}{u}\right)du. \qquad (6.24)$$

As is easily seen, putting in formula (6.23) $\tau = 0$ in view of the value of the Macdonald function $K_{1/2}(x)$ we arrive to the identity

$$\frac{e^{-x-y}}{\sqrt{xy}} = \frac{1}{2\sqrt{2\pi}}\int_0^\infty \exp\left(-\frac{1}{2}\left(\frac{xy}{u}+\frac{xu}{y}+\frac{uy}{x}+2u\right)\right)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{u}\right)\frac{du}{\sqrt{u}}. \quad (6.25)$$

Finally in this section we demonstrate some index integrals being corollaries of the above representations. For instance, by virtue of the inversion of the cosine and the sine Fourier transforms one can deduce from (6.5)-(6.6) the following formulae

$$\int_0^\infty \Re K_{\alpha+iy}(x)\cos(y\tau)dy = \frac{\pi}{2}e^{-x\cosh\tau}\cosh(\alpha\tau),\ x>0, \tau\geq 0, \qquad (6.26)$$

$$\int_0^\infty \Im K_{\alpha+iy}(x)\sin(y\tau)dy = \frac{\pi}{2}e^{-x\cosh\tau}\sinh(\alpha\tau),\ x>0,\tau\geq 0. \tag{6.27}$$

Furthermore, direct computations with the Fubini theorem by virtue of formulae (6.23)-(6.24) enable us to obtain the equalities (see formulae 2.16.55.2 in Prudnikov et al. [2])

$$\int_0^\infty \cos(a\tau)\Re K_{1/2+i\tau}(x)\Re K_{1/2+i\tau}(y)d\tau$$

$$= \frac{\pi}{4}\cosh\frac{a}{2}\left[\frac{x+y}{\sqrt{x^2+y^2+2xy\cosh a}}K_1\left(\sqrt{x^2+y^2+2xy\cosh a}\right)\right.$$

$$\left.+K_0\left(\sqrt{x^2+y^2+2xy\cosh a}\right)\right], \tag{6.28}$$

$$\int_0^\infty \cos(a\tau)\Im K_{1/2+i\tau}(x)\Im K_{1/2+i\tau}(y)d\tau$$

$$= \frac{\pi}{4}\cosh\frac{a}{2}\left[\frac{x+y}{\sqrt{x^2+y^2+2xy\cosh a}}K_1\left(\sqrt{x^2+y^2+2xy\cosh a}\right)\right.$$

$$\left.-K_0\left(\sqrt{x^2+y^2+2xy\cosh a}\right)\right]. \tag{6.29}$$

As the corollary of above formulae (6.28)-(6.29) we give below the following one

$$\int_0^\infty \cos(a\tau)\Re\left[K_{1/2+i\tau}(x)K_{1/2+i\tau}(y)\right]d\tau$$

$$= \frac{\pi}{2}\cosh\frac{a}{2}K_0\left(\sqrt{x^2+y^2+2xy\cosh a}\right). \tag{6.30}$$

6.2 The Lebedev-Skalskaya transforms

Our purpose here is to establish the mapping properties and inversions of Lebedev-Skalskaya transforms (6.1)-(6.2) in the weighted space $L_{\nu,p}(\mathbf{R}_+)$. In particular, in the case $\alpha = 1/2$ one can illustrate suitable inversion formulae and the Parseval-Plancherel identities. However, we begin from the case of a general α to give theorems analogously to Chapter 2 where we studied in detail the Kontorovich-Lebedev transform. Assume that $\tau \in \mathbf{R}_+$.

Theorem 6.1. *The operators of the Lebedev-Skalskaya transforms* (6.1) – (6.2) *map the space $L_{\nu,p}(\mathbf{R}_+)$, where $p \geq 1$, $\nu < 1-|\alpha|$ into the space $L_r(\mathbf{R}_+)$, $r \geq 1$ and parameters p, r in general have no dependence. Furthermore, functions $[\Re f](\tau)$, $[\Im f](\tau)$ are infinitely differentiable ones for any $f \in L_{\nu,p}(\mathbf{R}_+)$ with the parameters ν and p given above.*

Proof. The proof is straightforward by using the Hölder inequality and obtained estimate (6.10). Indeed, we have inequalities

$$\left|\left[\begin{Bmatrix}\Re\\ \Im\end{Bmatrix} f\right](\tau)\right| \le \int_0^\infty \left|\begin{Bmatrix}\Re\\ \Im\end{Bmatrix} K_{\alpha+i\tau}(t)f(t)\right| dt$$

$$\le e^{-\delta\tau}\left(\int_0^\infty K_\alpha^q(y\cos\delta)y^{(1-\nu)q-1}dy\right)^{1/q}||f||_{\nu,p}$$

$$= C_{\nu,\delta,\alpha,q}e^{-\delta\tau}||f||_{\nu,p},\ q = p/(p-1), \tag{6.31}$$

where $C_{\nu,\delta,\alpha,q}$ is a constant provided that $\nu < 1 - |\alpha|$ (in view of the asymptotic behavior of the Macdonald function $K_\alpha(x\cos\delta)$) and $\delta \in (0, \pi/2)$. Consequently, clearly that L_r-norms of Lebedev-Skalskaya transforms (6.1)-(6.2) can be immediately estimated as follows

$$\left|\left|\left[\begin{Bmatrix}\Re\\ \Im\end{Bmatrix} f\right]\right|\right|_r = \left(\int_0^\infty \left|\left[\begin{Bmatrix}\Re\\ \Im\end{Bmatrix} f\right](\tau)\right|^r d\tau\right)^{1/r}$$

$$\le C_{\nu,\delta,\alpha,q}||f||_{\nu,p}\left(\int_0^\infty e^{-r\delta\tau}d\tau\right)^{1/r} = C||f||_{\nu,p}, \tag{6.32}$$

where C is an absolute positive constant.

Further, appealing to (6.8)-(6.9) it is easy to show a performance of the differentiability by $\tau \ge 0$ under integral signs in view of the uniform convergence of the obtained integrals. Precisely, we find that for $k = 0, 1, \ldots,$

$$\frac{\partial^k}{\partial\tau^k}\left(\begin{Bmatrix}\Re\\ \Im\end{Bmatrix} K_{\alpha+i\tau}(x)\right) = \frac{1}{2}\int_{-\infty}^\infty e^{-x\cosh v + i\tau v}(iv)^k \begin{Bmatrix}\cosh(\alpha v)\\ \sinh(\alpha v)\end{Bmatrix} dv. \tag{6.33}$$

Thus,

$$\left|\frac{\partial^k}{\partial\tau^k}\left(\begin{Bmatrix}\Re\\ \Im\end{Bmatrix} K_{\alpha+i\tau}(x)\right)\right| \le \int_0^\infty e^{-x\cosh v}v^k \begin{Bmatrix}\cosh(\alpha v)\\ \sinh(\alpha v)\end{Bmatrix} dv. \tag{6.34}$$

Hence for the Lebedev-Skalskaya operators (6.1)-(6.2) it follows that

$$\frac{d^k}{d\tau^k}\left[\begin{Bmatrix}\Re\\ \Im\end{Bmatrix} f\right](\tau) = \int_0^\infty \frac{\partial^k}{\partial\tau^k}\left(\begin{Bmatrix}\Re\\ \Im\end{Bmatrix} K_{\alpha+i\tau}(t)\right) f(t)dt, \tag{6.35}$$

and therefore, recalling the Hölder inequality we continue

$$\left|\frac{d^k}{d\tau^k}\left[\begin{Bmatrix}\Re\\ \Im\end{Bmatrix} f\right](\tau)\right| \le \int_0^\infty \left|\frac{\partial^k}{\partial\tau^k}\left(\begin{Bmatrix}\Re\\ \Im\end{Bmatrix} K_{\alpha+i\tau}(t)\right) f(t)\right| dt$$

$$\le ||f||_{\nu,p}(\mathbf{R}_+)\left(\int_0^\infty \left|\frac{\partial^k}{\partial\tau^k}\left(\begin{Bmatrix}\Re\\ \Im\end{Bmatrix} K_{\alpha+i\tau}(y)\right)\right|^q y^{(1-\nu)q-1}dy\right)^{1/q}. \tag{6.36}$$

One can show that the integral at the right-hand side of equality (6.36) is a constant being uniformly estimated by $\tau \in \mathbf{R}_+$ by using the generalized Minkowski inequality (1.10). Namely, invoking with (6.34) it can be treated as follows

$$\left(\int_0^\infty \left|\frac{\partial^k}{\partial\tau^k}\left(\begin{Bmatrix}\Re\\ \Im\end{Bmatrix} K_{\alpha+i\tau}(y)\right)\right|^q y^{(1-\nu)q-1}dy\right)^{1/q}$$

$$\leq \int_0^\infty v^k \left\{ \begin{matrix} \cosh(\alpha v) \\ \sinh(\alpha v) \end{matrix} \right\} \left(\int_0^\infty e^{-qy\cosh v} y^{(1-\nu)q-1} dy \right)^{1/q} dv$$

$$= q^{\nu-1}\Gamma^{1/q}(q(1-\nu)) \int_0^\infty \frac{v^k}{\cosh^{1-\nu} v} \left\{ \begin{matrix} \cosh(\alpha v) \\ \sinh(\alpha v) \end{matrix} \right\} dv = C_{\nu,\alpha,k} < \infty, \tag{6.37}$$

where $C_{\nu,\alpha,k}$ is a constant. Indeed, one can easily verify under condition $\nu < 1 - |\alpha|$ that it provides the convergence of the integral by v in (6.37). This completes the proof of Theorem 6.1. •

Let us define the range of the Lebedev-Skalskaya index transforms of $L_{\nu,p}$-functions similarly to space (2.13) by

$$LS^\alpha(L_{\nu,p}) = \left\{ g : \; g(\tau) = \left[\left\{ \begin{matrix} \Re \\ \Im \end{matrix} \right\} f \right](\tau), \; f \in L_{\nu,p}(\mathbf{R}_+) \right\}, \; p \geq 1, \; \nu < 1 - |\alpha|. \tag{6.38}$$

In order to establish inversions of the Lebedev-Skalskaya operators in suitable form of the approximation operator like (2.14) by means of the representations obtained above attract our attention to the case $\alpha = 1/2$. Without loss of generality consider now in particular, $\Re$-transform (6.1). To describe the range (6.38) in this case define the approximation operator such that

$$\left(I_\varepsilon^{\Re} g \right)(x) = \frac{4x^\varepsilon}{\pi^2} \int_0^\infty \cosh((\pi - 2\varepsilon)\tau) \Re K_{1/2+i\tau}(x) g(\tau) d\tau, \tag{6.39}$$

where $\varepsilon \in (0, \pi/2), \; x > 0$.

Theorem 6.2. *On functions $g(\tau) = [\Re f](\tau)$ being represented by the Lebedev-Skalskaya transform* (6.1) *with $\alpha = 1/2$ and $f \in L_{\nu,p}(\mathbf{R}_+), \; 1 \leq p \leq \infty, \; \nu < 1/2$, operator* (6.39) *has the following form*

$$\left(I_\varepsilon^{\Re} g \right)(x) = \frac{x^\varepsilon \sin \varepsilon}{\pi} \left[\int_0^\infty K_0 \left(\sqrt{x^2 + y^2 - 2xy\cos(2\varepsilon)} \right) f(y) dy \right.$$

$$\left. + \int_0^\infty \frac{(x+y) K_1 \left(\sqrt{x^2 + y^2 - 2xy\cos(2\varepsilon)} \right)}{\sqrt{x^2 + y^2 - 2xy\cos(2\epsilon)}} f(y) dy \right], \; x > 0. \tag{6.40}$$

Proof. First by virtue of (6.11) and (6.31) observe that for each $\varepsilon \in (0, \pi/2)$ one can deduce the estimate, namely

$$\left| \left(I_\varepsilon^{\Re} g \right)(x) \right| \leq \frac{2x^{\varepsilon - 1/2}}{\pi \cos\delta} e^{-x\cos\delta}$$

$$\times \int_0^\infty \cosh((\pi - 2\varepsilon)\tau) e^{-2\delta\tau} d\tau$$

$$\times \int_0^\infty e^{-y\cos\delta} |f(y)| \frac{dy}{\sqrt{y}} \leq C_{\delta,\varepsilon} x^{\varepsilon - 1/2} e^{-x\cos\delta} ||f||_{\nu,p}, \tag{6.41}$$

where
$$C_{\delta,\varepsilon} = \frac{2}{\pi \cos\delta} \int_0^\infty \cosh((\pi - 2\varepsilon)\tau) e^{-2\delta\tau} d\tau$$
$$\times \left(\int_0^\infty e^{-yq\cos\delta} y^{(1/2-\nu)q-1} dy \right)^{1/q} < \infty \tag{6.42}$$
owing to the conditions $\nu < 1/2$ and $\pi/2 - \varepsilon < \delta < \pi/2$. Consequently, substituting the value of $g(\tau)$ being given by formula
$$g(\tau) = \int_0^\infty \Re K_{1/2+i\tau}(y) f(y) dy, \tag{6.43}$$
in (6.39) change the order of integration appealing to the Fubini theorem in view of estimate (6.42). Finally, call representation (6.28) which leads us to the desired formula (6.40). Theorem 6.2 is proved. •

Turn now to the inversion of the Lebedev-Skalskaya transform (6.43) of $L_{\nu,p}$-functions in terms of the approximation operator (6.39).

Theorem 6.3. *Let $g(\tau)$ be given by formula* (6.43) *, where $f \in L_{\nu,p}(\mathbf{R}_+)$, $p \geq 1$, $-1 < \nu < 1/2$. Then*
$$f(x) = \left(I^{\Re} g \right)(x), \tag{6.44}$$
where $\left(I^{\Re} g\right)(x)$ is understood as
$$\left(I^{\Re} g \right)(x) = \text{l.i.m.}_{\varepsilon \to 0+} \left(I^{\Re}_{\varepsilon} g \right)(x),\ x > 0, \tag{6.45}$$
by the $L_{\nu,p}$-norm (1.19)*. In addition, the limit in* (6.45) *exists almost everywhere on* $\mathbf{R}_+$.

Proof. The proof of this theorem shall be completed in the same manner as for example in Theorem 2.2. Denoting by
$$\hat{R}(x,t,\varepsilon) = x^{\varepsilon+1} \sin\varepsilon \Big[\sin 2\varepsilon (t^2+1) K_0 \left(x \sin 2\varepsilon (t^2+1)^{1/2} \right)$$
$$+ (1 + \cos 2\varepsilon + t \sin 2\varepsilon)(t^2+1)^{1/2} K_1 (x \sin 2\varepsilon (t^2+1)^{1/2}) \Big],\ t \geq -\cot 2\varepsilon, \tag{6.46}$$
$$\hat{R}(x,t,\varepsilon) = 0,\ t < -\cot 2\varepsilon, \tag{6.47}$$
interchange variable in (6.40) by the replacement $y = x(\cos 2\varepsilon + t \sin 2\varepsilon)$. Hence we obtain for $\varepsilon \in (0, \pi/2)$ that
$$\left(I^{\Re}_{\varepsilon} g \right)(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{1}{t^2+1} f(x(\cos 2\varepsilon + t \sin 2\varepsilon)) \hat{R}(x,t,\varepsilon) dt. \tag{6.48}$$

Meanwhile, it is not difficult to see, that due to formulae (1.96)-(1.97) of the asymptotic behavior of Macdonald's functions $K_0(z)$, $K_1(z)$, for any $t \in \mathbf{R}$, $x \in \mathbf{R}_+$ and $\varepsilon \in (0, \pi/2)$ it follows that $|\hat{R}(x,t,\varepsilon)| < C$, where C is a positive constant and
$$\lim_{\varepsilon \to 0+} \hat{R}(x,t,\varepsilon) = 1. \tag{6.49}$$

Further, owing to the generalized Minkowski inequality (1.10) estimate $L_{\nu,p}$-norm of the operator in the left hand-side of (6.48). We have

$$\left\|\left(I_\varepsilon^{\Re} g\right)\right\|_{\nu,p} \leq \frac{1}{\pi}\int_{-\cot 2\varepsilon}^{\infty}\frac{1}{t^2+1}\|f(x(\cos 2\varepsilon+t\sin 2\varepsilon))\hat{R}(x,t,\varepsilon)\|_{\nu,p}dt$$

$$\leq C\int_{-\cot 2\varepsilon}^{\infty}\frac{1}{t^2+1}\|f(x(\cos 2\varepsilon+t\sin 2\varepsilon))\|_{\nu,p}dt$$

$$= C\|f\|_{\nu,p}\int_{-\cot 2\varepsilon}^{\infty}\frac{(\cos 2\varepsilon+t\sin 2\varepsilon)^{-\nu}}{t^2+1}dt = C_1\|f\|_{\nu,p}, \tag{6.50}$$

where constants are provided by the boundedness of the function $\hat{R}(x,t,\varepsilon)$ and uniformly convergent integral (5.116) under $-1<\nu<1/2$. Therefore, making use properties of the Poisson kernel (1.14), Theorem 1.4 for $L_{\nu,p}$-spaces, continuity of L_p-norm and limit relation (6.49) immediately conclude that

$$\left\|\left(I_\varepsilon^{\Re} g\right)-f\right\|_{\nu,p} \leq \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{t^2+1}\|f(x(\cos 2\varepsilon+t\sin 2\varepsilon))\hat{R}(x,t,\varepsilon)$$

$$-f(x)\|_{\nu,p}dt \to 0,\ \varepsilon\to 0+. \tag{6.51}$$

The existence of the limit almost everywhere on $\mathbf{R}_+$ one can motivate by Theorem 1.3. This ends the proof of Theorem 6.3. •

As a corollary estimate (6.50) gives the uniform by ε inequality of kind

$$\left\|\left(I_\varepsilon^{\Re} g\right)\right\|_{\nu,p} \leq C\left\|\left(I^{\Re} g\right)\right\|_{\nu,p} \tag{6.52}$$

for any $g\in LS^{1/2}(L_{\nu,p})$, $p\geq 1$, $-1<\nu<1/2$. Consequently, Theorem 6.3 implies that

$$[\Re f](\tau)\equiv 0,\ f\in L_{\nu,p}(\mathbf{R}_+),\ p\geq 1, -1<\nu<1/2, \tag{6.53}$$

if and only if $f(x)=0$ almost everywhere on $\mathbf{R}_+$. Furthermore, one can introduce a norm in the space $LS^{1/2}(L_{\nu,p})$ for instance, by the equality

$$\|g\|_{LS^{1/2}(L_{\nu,p})} = \|f\|_{\nu,p},\ g=[\Re f](\tau). \tag{6.54}$$

Let us prove next theorem to obtain a description of the range (6.38).

Theorem 6.4. *An arbitrary function $g(\tau)$ being defined on $\mathbf{R}_+$ belongs to the space $LS^{1/2}(L_{\nu,p})$, $p\geq 1, -1<\nu<1/2$, if and only if $g(\tau)\in L_r(\mathbf{R}_+)$, where $1\leq r\leq\infty$ in the necessity part, $2<r<\infty$ in the sufficiency part and also*

$$\text{l.i.m.}_{\varepsilon\to 0+}\left(I_\varepsilon^{\Re} g\right)\in L_{\nu,p}(\mathbf{R}_+). \tag{6.55}$$

Proof. The necessity of condition (6.55) follows directly from Theorem 6.3 and estimate (6.32). Let us prove the sufficiency. Assume that $g(\tau)\in L_r(\mathbf{R}_+)$, $2<r<\infty$

and relation (6.55) holds. Our purpose is to show that in this case there is a function $f \in L_{\nu,p}$ under conditions on parameters ν, p in the theorem such that the equality

$$g = [\Re f] \tag{6.56}$$

takes place. As is easy to conclude from condition (6.55) that for enough small $\varepsilon > 0$ the function $\left(I_\varepsilon^{\Re} g\right)$ belongs to $L_{\nu,p}(\mathbf{R}_+)$, $p \geq 1, -1 < \nu < 1/2$. Moreover, one can calculate the following composition of operators (see also (6.43))

$$\left[\Re\left(I_\varepsilon^{\Re} g\right)\right](\tau) = \int_0^\infty \Re K_{1/2+i\tau}(y)\left(I_\varepsilon^{\Re} g\right)(y)dy. \tag{6.57}$$

As usually for a set of smooth functions g with a compact support on $\mathbf{R}_+$, which is dense in L_r, substitute the expression of the operator $\left(I_\varepsilon^{\Re}\right)$ given by formula (6.39) in (6.57) and change the order of integration. The inner integral arises, namely

$$I(\tau, t) = \int_0^\infty y^\varepsilon \Re K_{1/2+i\tau}(y) \Re K_{1/2+it}(y)dy, \tag{6.58}$$

and it can be evaluated by formula 2.16.33.2 in Prudnikov et al. [2] (see also (2.30)). Hence we obtain

$$I(\tau, t) = \frac{2^{\varepsilon-2}}{\Gamma(\varepsilon)}\left[\frac{1}{\varepsilon^2 + (\tau+t)^2}\left|\Gamma\left(\frac{\varepsilon + i(t+\tau)}{2} + 1\right)\Gamma\left(\frac{\varepsilon + 1 + i(\tau - t)}{2}\right)\right|^2\right.$$
$$\left. + \frac{1}{\varepsilon^2 + (\tau - t)^2}\left|\Gamma\left(\frac{\varepsilon + i(\tau - t)}{2} + 1\right)\Gamma\left(\frac{\varepsilon + 1 + i(\tau + t)}{2}\right)\right|^2\right]. \tag{6.59}$$

Consequently, one can write composition (6.57) as follows

$$\left[\Re\left(I_\varepsilon^{\Re} g\right)\right](\tau) = g_\varepsilon(\tau) = \frac{4}{\pi^2}\int_0^\infty \cosh((\pi - 2\varepsilon)t) I(\tau, t) g(t)dt, \tag{6.60}$$

for $\varepsilon \in (0, \pi/2)$. In order to spread equality (6.60) for all $g \in L_r$, we may prove the boundedness of the operator in the right-hand side of (6.60). However, owing to (6.59) and Stirling's formula (1.33) the kernel of the integrand there equals to

$$O\left(e^{-2\varepsilon t + \pi[(t-\tau) - |t-\tau|]/2}\right), \ (t, \tau) \in \mathbf{R}_+ \times \mathbf{R}_+, \ \varepsilon \in (0, \pi/2). \tag{6.61}$$

One can establish the estimate like (2.33), namely, according to the Hölder inequality (1.8) there exists some parameter $\delta \in (\pi/2 - 2\varepsilon, \pi/2)$ and a constant C such that

$$\left|\left[\Re\left(I_\varepsilon^{\Re} g\right)\right](\tau)\right| \leq Ce^{(\delta - \pi/2)\tau}\int_0^\infty e^{(\pi/2 - 2\varepsilon - \delta)t} g(t)dt \leq qC_\varepsilon ||g||_r. \tag{6.62}$$

Thus we find the boundedness of the operator in the right-hand side of (6.60) in the space L_r, $2 < r < \infty$. Hence invoking with (6.59) and simple interchanges of variables one can represent the function $g_\varepsilon(\tau)$ as

$$g_\varepsilon(\tau) = \frac{2^\varepsilon}{\Gamma(\varepsilon + 1)\pi^2}\int_{\tau/\varepsilon}^\infty \frac{g(\varepsilon v - \tau)}{v^2 + 1}$$

$$\times \cosh((\pi - 2\varepsilon)(\varepsilon v - \tau)) \left| \Gamma\left(\varepsilon\frac{(1+iv)}{2} + 1\right) \Gamma\left(i\tau + \frac{1}{2} + \frac{\varepsilon(1-iv)}{2}\right) \right|^2 dv$$

$$+ \frac{2^\varepsilon}{\Gamma(\varepsilon+1)\pi^2} \int_{-\infty}^{\tau/\varepsilon} \frac{g(\tau - \varepsilon v)}{v^2+1}$$

$$\times \cosh((\pi - 2\varepsilon)(\tau - \varepsilon v)) \left| \Gamma\left(\varepsilon\frac{(1+iv)}{2} + 1\right) \Gamma\left(i\tau + \frac{1}{2} + \frac{\varepsilon(1-iv)}{2}\right) \right|^2 dv$$

$$= g_{1\varepsilon}(\tau) + g_{2\varepsilon}(\tau). \tag{6.63}$$

Let us estimate the absolute value of $g_{1\varepsilon}(\tau)$ depending on variable $\tau \geq 0$. Indeed, owing to (6.61) we find

$$|g_{1\varepsilon}(\tau)| \leq C\varepsilon \int_0^\infty \frac{|g(t)|}{(\tau+t)^2 + \varepsilon^2} dt. \tag{6.64}$$

Making use the Hölder inequality (1.8) and taking $\tau \geq 1$ arrive to the following relations

$$|g_{1\varepsilon}(\tau)| \leq C\varepsilon ||g||_r \left(\int_0^\infty \frac{1}{((\tau+t)^2+\varepsilon^2)^{r'}}\right)^{1/r'} dt$$

$$\leq C_1 \frac{\varepsilon}{\tau^{1+1/r}} ||g||_r, \; r' = r/(r-1), \; r < \infty. \tag{6.65}$$

Meanwhile, for $0 \leq \tau < 1$ we set

$$|g_{1\varepsilon}(\tau)| \leq C\varepsilon \left(\int_0^1 \frac{|g(t)|}{t^2+\varepsilon^2} dt + \int_1^\infty \frac{|g(t)|}{t^2+\varepsilon^2} dt\right)$$

$$= C\varepsilon \left(I_1(\varepsilon) + I_2(\varepsilon)\right). \tag{6.66}$$

Hence for the integral I_2 we obtain

$$I_2(\varepsilon) \leq ||g||_r \left(\int_1^\infty \frac{dt}{t^{2r'}}\right)^{1/r'} < \infty. \tag{6.67}$$

Conversely, the integral I_1 can be estimated as follows

$$\varepsilon I_1(\varepsilon) = \varepsilon \int_0^1 \frac{|g(t)|}{t^2+\varepsilon^2} dt = \int_0^\varepsilon \frac{|g(t\varepsilon)|}{t^2+1} dt$$

$$\leq ||g||_r \varepsilon^{-1/r} \left(\int_0^\varepsilon \frac{dt}{(t^2+1)^{r'}}\right)^{1/r'}$$

$$\leq \varepsilon^{1-2/r} ||g||_r \to 0, \; \varepsilon \to 0+, r > 2. \tag{6.68}$$

Combining results in (6.67)-(6.68) find that under $2 < r < \infty$ the function $g_{1\varepsilon}(\tau) \in L_r(\mathbf{R}_+)$ and moreover,

$$||g_{1\varepsilon}||_r \to 0, \; \varepsilon \to 0+. \tag{6.69}$$

Concerning the function from (6.63) one can write

$$g_{2\varepsilon}(\tau) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{g(\tau-\varepsilon v)}{v^2+1} H\left(\frac{\tau}{\varepsilon} - v\right) \hat{h}(\tau, v, \varepsilon) dv, \tag{6.70}$$

we $H(x)$ is the Heaviside function and $\hat{h}(\tau, v, \varepsilon)$ defined by formula

$$\hat{h}(\tau, v, \varepsilon) = \frac{2^{\varepsilon}}{\Gamma(\varepsilon+1)\pi} \cosh((\pi - 2\varepsilon)(\tau - \varepsilon v))$$
$$\times \left|\Gamma\left(\varepsilon\frac{(1+iv)}{2} + 1\right)\Gamma\left(i\tau + \frac{1}{2} + \frac{\varepsilon(1-iv)}{2}\right)\right|^2. \tag{6.71}$$

Taking into account the above estimates and conditions of this theorem invoke with the generalized Minkowski inequality (1.10) and supplement formula (1.29) for gamma-functions. Hence from equality (6.63) we obtain

$$||g_\varepsilon - g||_r \leq ||g_{1\varepsilon}||_r + ||g_{2\varepsilon} - g||_r$$
$$\leq ||g_{1\varepsilon}||_r + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{1}{v^2+1}$$
$$\times \left|\left|g(\tau - \varepsilon v) H\left(\frac{\tau}{\varepsilon} - v\right) \hat{h}(\tau, v, \varepsilon) - g(\tau)\right|\right|_r dv \to 0, \ \varepsilon \to 0, \ 2 < r < \infty. \tag{6.72}$$

However, on the other hand, estimate (6.31) implies that the operator $[\Re f]$ is bounded in $L_{\nu,p}$, $p \geq 1, -1 < \nu < 1/2$. In other words, there exists the following limit in $L_{\nu,p}$-norm (1.19)

$$\text{l.i.m.}_{\varepsilon\to 0+}\left[\Re\left(I_\varepsilon^{\Re} g\right)\right] = \left[\Re \text{l.i.m.}_{\varepsilon\to 0+}\left(I_\varepsilon^{\Re} g\right)\right]$$
$$= [\Re f], \ f = \left(I^{\Re} g\right) \in L_{\nu,p}(\mathbf{R}_+). \tag{6.73}$$

Since the composition $\left[\Re\left(I_\varepsilon^{\Re} g\right)\right]$ converges in the L_r-norm too, the limit functions coincide almost everywhere on $\mathbf{R}_+$. Thus from equality (6.73) we lead to (6.56). Theorem 6.4 is completely proved. •

Remark 6.1. In the same manner one can treat the $\Im$-transform (6.2) in the case $\alpha = 1/2$. Thus we obtained pairs of reciprocal formulae for the Lebedev-Skalskaya $\Re$-, $\Im$-transforms such that

$$[\Re f](\tau) = \int_0^\infty \Re K_{1/2+i\tau}(x) f(x) dx, \tag{6.74}$$

$$f(x) = \frac{4}{\pi^2}\int_0^\infty \cosh(\pi\tau) \Re K_{1/2+i\tau}(x)[\Re f](\tau) d\tau, \tag{6.75}$$

$$[\Im f](\tau) = \int_0^\infty \Im K_{1/2+i\tau}(x) f(x) dx, \tag{6.76}$$

$$f(x) = \frac{4}{\pi^2}\int_0^\infty \cosh(\pi\tau) \Im K_{1/2+i\tau}(x)[\Im f](\tau) d\tau, \tag{6.77}$$

where we mean $f \in L_{\nu,p}(\mathbf{R}_+)$ for $1 \leq p \leq \infty$, $-1 < \nu < 1/2$ and integrals (6.74), (6.76) converge absolutely. Meanwhile, the convergence of integrals (6.75), (6.77) are meant in terms of the corresponding approximation operators (6.39) $\left(I_\varepsilon^{\Re}\right)$, $\left(I_\varepsilon^{\Im}\right)$ by respective Theorems 6.3-6.4.

6.3 L_2-theory of the Lebedev-Skalskaya transforms

We are going to consider here the limit case of the range (6.38) $LS^{1/2}(L_{\nu,p})$ for the Lebedev-Skalskaya transforms (6.74), (6.76), namely when $\nu = 1/2$ and $p = 2$. In other words, we take f from the space $L_2(\mathbf{R}_+) \equiv L_{1/2,2}(\mathbf{R}_+)$. Attracting our attention to the $\Re$-transform we construct the Plancherel theory similarly as for the index transforms considered above.

In accordance with formula (2.43) let us define the $\Re$-transform (6.74) in the form

$$[\Re f](\tau) = \text{l.i.m.}_{N\to\infty} \int_{1/N}^{N} \Re K_{1/2+i\tau}(y) f(y)dy, \tag{6.78}$$

where $f \in L_2(\mathbf{R}_+)$ and the limit in (6.78) is understood in the meaning of convergence in certain Hilbert space that shall be defined below. The integral in (6.78) converges absolutely owing to Theorem 6.1. Indeed, if $f(x) \in L_2(\mathbf{R}_+)$, then for any number $N > 0$ $f(x) \in L_2([1/N, N])$. Furthermore, the estimate

$$\int_{1/N}^{N} y^{2\nu-1}|f(y)|^2 dy < C \int_{1/N}^{N} |f(y)|^2 dy = C||f||^2_{L_2([1/N,N])} \tag{6.79}$$

enables us to conclude that $f(x) \in L_{\nu,2}([1/N, N])$ for any $-1 < \nu < 1/2$. Consequently, due to Theorem 6.1 integral (6.78) is absolutely convergent and the Lebedev-Skalskaya transform (6.78) of the function $f_N = f(x),\ x \in [1/N, N],\ f(x) = 0,\ 0 < x < 1/N$ exists.

Appealing to formula (6.75) determine the Hilbert space $L_2\left(\mathbf{R}_+; \frac{4}{\pi^2}\cosh(\pi\tau)\right)$ with the norm

$$||h||_{L_2\left(\mathbf{R}_+;\frac{4}{\pi^2}\cosh(\pi\tau)\right)} = \frac{2}{\pi}\left(\int_0^\infty \cosh(\pi\tau)|h(\tau)|^2 d\tau\right)^{1/2}. \tag{6.80}$$

Hence take the inner product in this space and write formally

$$\begin{aligned}([\Re f], [\Re g]) &= \frac{4}{\pi^2}\int_0^\infty \cosh(\pi\tau)[\Re f](\tau)\overline{[\Re g](\tau)}d\tau \\ &= \frac{4}{\pi^2}\int_0^\infty \cosh(\pi\tau)[\Re f](\tau)\overline{\int_0^\infty \Re K_{1/2+i\tau}(y)g(y)dy}d\tau \\ &= \int_0^\infty \overline{g(y)}dy\frac{4}{\pi^2}\int_0^\infty \cosh(\pi\tau)\Re K_{1/2+i\tau}(y)[\Re f](\tau)d\tau \\ &= \int_0^\infty f(y)\overline{g(y)}dy = \langle f, g\rangle_{L_2(\mathbf{R}_+)},\end{aligned} \tag{6.81}$$

where the notation $\langle , \rangle_{L_2}$ is fixed for the inner product in the space $L_2(\mathbf{R}_+)$. The respective conditions of validity of equalities (6.81) can be given by

Theorem 6.5. *If $g(x) \in L_1(\mathbf{R}_+; e^{-x\cos\delta}/\sqrt{x})$ and $[\Re f] \in L_1(\mathbf{R}_+; \exp((\pi - \delta)\tau))$, $\delta \in [0, \pi/2)$, then the Parseval equality for the Lebedev-Skalskaya transform (6.74) takes place*

$$([\Re f], [\Re g]) = \langle f, g\rangle. \tag{6.82}$$

In addition, for $f = g$ we obtain the isometrical identity in the form

$$||f||_2 = ||[\Re f]||_{L_2\left(\mathbf{R}_+; \frac{4}{\pi^2}\cosh(\pi\tau)\right)}. \tag{6.83}$$

Proof. The proof of this theorem immediately follows from estimate (6.11) and Fubini's theorem that allows us to change the order of integration in (6.81) by virtue of the absolute convergence of the respective iterated integral. •

Similarly as for the Kontorovich-Lebedev transform (2.43) one can verify the validity of the Parseval formula (6.82) for the space of smooth functions with compact support on $\mathbf{R}_+$. Although here it is enough to take functions from $C^{(1)}(\mathbf{R}_+)$. Nevertheless, we need to establish useful integral representation to estimate the kernel $\Re K_{1/2+i\tau}(x)$ by the index τ at infinity for each fixed $x > 0$. Indeed, according to (1.105)-(1.106) (see also Erdélyi et al. [1]) one can extend these formulae on the complex parameter $\mu = \Re\mu + i\tau$, $|\Re\mu| < 1$, $\tau \in \mathbf{R}$ in view of the uniform convergence by μ being verified with the aid of the replacement $v = \sinh u$ as well as Abel's test of the uniform convergence of integrals. Consequently, letting there $\mu = 1/2 + i\tau$ after simple manipulations it is not difficult to obtain the representation in the form

$$\cosh\left(\frac{\pi}{2}\right) \Re K_{1/2+i\tau}(x) = \frac{1}{\sqrt{2}} \int_0^\infty \cos(\tau u)$$

$$\times \left[\cos(x\sinh u)\cosh(u/2) + \sin(x\sinh u)\sinh(u/2)\right] du, \ x > 0. \tag{6.84}$$

Further, in the same manner integral (6.84) converges uniformly by $x \in [0, a]$, $a > 0$. Therefore, integrating through in (6.84) by x we find that

$$\cosh\left(\frac{\pi}{2}\right) \int_0^x \Re K_{1/2+i\tau}(t)dt = \frac{1}{2\sqrt{2}} \int_0^\infty \cos(\tau u)$$

$$\times \left[\frac{\sin(x\sinh u)}{\sinh(u/2)} + \frac{1 - \cos(x\sinh u)}{\cosh(u/2)}\right] du, \ x > 0. \tag{6.85}$$

Meanwhile, one can show that for each fixed $x > 0$ the function

$$\cosh(\pi\tau/2)|\Re K_{1/2+i\tau}(x)| < C$$

uniformly by $\tau \in \mathbf{R}_+$. Namely, to prove it consider separately each integral in (6.84), dividing it for two ones. First as is evident from the uniform convergence of the integral (6.84) by $\tau \in [0, A), A > 0$ the function $\cosh(\pi\tau/2)\Re K_{1/2+i\tau}(x)$ is continuous by τ for any $x > 0$. Thus we may to show its boundedness at infinity. For the first integral in (6.84) we have

$$I(\tau, x) = \int_0^\infty \cos(x \sinh u) \cosh(u/2) \cos(\tau u) du$$

$$= \left(\int_0^N + \int_N^\infty \right) \cos(x \sinh u) \cosh(u/2) \cos(\tau u) du$$

$$= I_1(\tau, x) + I_2(\tau, x), \tag{6.86}$$

where N is an enough big positive number. The integral $I_1(\tau, x)$ is absolutely convergent and consequently, it follows that $|I(\tau, x)| < C$. For the second integral in (6.86) we find with the interchange $v = \sinh u$ that

$$I_2(\tau, x) = O\left(\int_{N_1}^\infty \frac{\cos(xv) \cos\left(\tau \log(v + (v^2+1)^{1/2})\right)}{\sqrt{v}} dv \right)$$

$$= O\left(\int_{N_1}^\infty \frac{\cos\left(xv + \tau \log(v + (v^2+1)^{1/2})\right)}{\sqrt{v}} dv \right)$$

$$+O\left(\int_{N_1}^\infty \frac{\cos\left(xv - \tau \log(v + (v^2+1)^{1/2})\right)}{\sqrt{v}} dv \right) = O\left(I_{21}(\tau, x)\right) + O\left(I_{22}(\tau, x)\right). \tag{6.87}$$

Treat each integral in the right-hand side of (6.87) integrating by parts and considering enough big $\tau \in \mathbf{R}_+$. Then, for instance, we obtain

$$I_{21}(\tau, x) = \int_{N_1}^\infty \frac{\cos\left(xv + \tau \log(v + (v^2+1)^{1/2})\right)}{\sqrt{v}} dv$$

$$= -\frac{\sin\left(xN_1 + \tau \log(N_1 + (N_1^2+1)^{1/2})\right) (N_1^2+1)^{1/2}}{\sqrt{N_1}(x(N_1^2+1)^{1/2} + \tau)}$$

$$+\frac{1}{2} \int_{N_1}^\infty \frac{\sin\left(xv + \tau \log(v + (v^2+1)^{1/2})\right) (v^2+1)^{1/2}}{v^{3/2}(x(v^2+1)^{1/2} + \tau)} dv$$

$$-\int_{N_1}^\infty \frac{\sin\left(xv + \tau \log(v + (v^2+1)^{1/2})\right) \sqrt{v}(v^2+1)^{-1/2}}{(x(v^2+1)^{1/2} + \tau)} dv$$

$$+x \int_{N_1}^\infty \frac{\sin\left(xv + \tau \log(v + (v^2+1)^{1/2})\right) \sqrt{v}}{(x(v^2+1)^{1/2} + \tau)^2} dv. \tag{6.88}$$

Hence is obvious to observe that for each $x > 0$ the right-hand side of (6.88) is bounded by τ and we have

$$|I_{21}(\tau, x)| \le \frac{1}{\sqrt{N_1}} + C_x \int_{N_1}^\infty \frac{dv}{v^{3/2}}, \tag{6.89}$$

where C_x is a constant that does not depend from τ. So we obtained that $I_{21}(\tau, x) = O(1)$, $\tau \to \infty$. Similarly verify the integral $I_{22}(\tau, x)$ in (6.87), integrating by parts and choosing finally N_1 enough big number to separate a denominator of the integrand from possible zeros in this case. Then details of this verification as well as the estimation of the second integral in (6.84) we leave to the reader. Thus conclude our desired assumption concerning the uniform boundedness of the function $\cosh(\pi\tau/2)\Re K_{1/2+i\tau}(x)$.

Theorem 6.6. *If the function $f(x) \in C^{(1)}(\mathbf{R}_+)$, then the Lebedev-Skalskaya transform* (6.78) *belongs to the space $L_1(\mathbf{R}_+; \exp(\pi\tau/2))$.*

Proof. It is a simple matter to check that integral (6.78) is absolutely convergent and under conditions of the theorem defines a continuous function of the variable $\tau \in \mathbf{R}_+$. Moreover, one can write multiplying through in (6.78) by $\cosh(\pi\tau/2)$ that

$$\cosh(\pi\tau/2)[\Re f](\tau) = \cosh(\pi\tau/2) \int_0^\infty \Re K_{1/2+i\tau}(y) f(y) dy$$

$$= \cosh(\pi\tau/2) \int_I \Re K_{1/2+i\tau}(y) f(y) dy, \tag{6.90}$$

where we denoted by I the support of a function f being a compact set on $\mathbf{R}_+$. Hence integrating by parts in (6.90) it becomes

$$\cosh(\pi\tau/2)[\Re f](\tau) = -\cosh(\pi\tau/2) \int_0^\infty f'(y) \int_0^y \Re K_{1/2+i\tau}(t) dt. \tag{6.91}$$

Now calling representation (6.85) substitute it into the right-hand side of (6.91) and change the order of integration by the Fubini theorem according to an absolute convergence of the iterated integral. Denoting by

$$F_1(u) = \frac{1}{2\sqrt{2}\sinh(u/2)} \int_{I_1} f'(y) \sin(y \sinh u) dy, \tag{6.92}$$

$$F_2(u) = \frac{1}{2\sqrt{2}\cosh(u/2)} \int_{I_1} f'(y)(1 - \cos(y \sinh u)) dy, \tag{6.93}$$

where we mean by I_1 the support of the function f' arrive to the equality of type

$$\cosh(\pi\tau/2)[\Re f](\tau) = -\int_0^\infty [F_1(u) + F_2(u)] \cos(\tau u) du. \tag{6.94}$$

Hence one can achieve the order of integrability by τ at infinity of the function $\cosh(\pi\tau/2)[\Re f](\tau)$ taking into account differential properties of functions $F_i(u)$, $i = 1, 2$ and their vanishing with derivatives, when u tends to zero or infinity. Thus twice integration by parts leads to the representation

$$\cosh(\pi\tau/2)[\Re f](\tau) = \frac{1}{\tau^2} \int_0^\infty [F_1''(u) + F_2''(u)] \cos(\tau u) du. \tag{6.95}$$

Therefore, owing to an absolute and uniform by $\tau \in \mathbf{R}_+$ convergence of the integral at the right-hand side of (6.95) being verified directly by a summable majorant we obtain that it equals $O(1/\tau^2)$, $\tau \to \infty$. This ends the proof of Theorem 6.6. •

Theorem 6.7. *For the functions $f(x)$ from the space $C^{(1)}(\mathbf{R}_+)$ the Parseval equality* (6.82) *is true.*

Proof. As it straightforwards from the previous theorem and Theorem 6.4 for functions $f \in C^{(1)}(\mathbf{R}_+)$ formula (6.75) holds. Besides integral (6.75) is absolutely convergent owing to the above estimates and one can derive the series of equalities (6.81), taking into account the compactness of a support of the function $f(x)$ and by using the Fubini theorem. •

As for the Kontorovich-Lebedev transform in Chapter 2 choose an arbitrary function from $L_2(\mathbf{R}_+)$. As is known there exists some sequence of functions with the compact support from the space $C^{(1)}(\mathbf{R}_+)$ being convergent to the given function f by norm of the space $L_2(\mathbf{R}_+)$. Denoting as usually through f_n the common term of this sequence from Theorem 6.7 we deduce the equality

$$\int_0^\infty |f_n(x) - f_m(x)|^2 dx = \frac{4}{\pi^2}\int_0^\infty \cosh(\pi\tau)|[\Re f_n](\tau) - [\Re f_m](\tau)|^2 d\tau. \tag{6.96}$$

The left-hand side of equality (6.96) tends to zero by $m, n \to \infty$ by virtue of the completeness of $L_2(\mathbf{R}_+)$. Consequently, the sequence $\{[\Re f_n](\tau)\}$ is the Cauchy sequence. Furthermore, there exists a function $h(\tau) \equiv [\Re f](\tau) \in L_2\left(\mathbf{R}_+; \frac{4}{\pi^2}\cosh(\pi\tau)\right)$ such that $[\Re f_n](\tau) \to h(\tau)$ by the norm of this space. As is clear

$$[\Re f_n](\tau) = \int_{I_n} \Re K_{1/2+i\tau}(y) f_n(y) dy, \tag{6.97}$$

where I_n is meant the least segment being contained the support of the function f_n. Hence integrating through in (6.97) by τ we find

$$\int_0^\tau [\Re f_n](t) dt = \int_{I_n} f_n(y) dy \int_0^\tau \Re K_{1/2+it}(y) dt$$

$$= \int_0^\infty \Re K(\tau, y) f_n(y) dy, \tag{6.98}$$

where we put by

$$\Re K(\tau, y) = \int_0^\tau \Re K_{1/2+it}(y) dt. \tag{6.99}$$

Turn now to the left-hand side of the equality (6.98). In fact, since $[\Re f_n](t)$ belongs to $L_2\left(\mathbf{R}_+; \frac{4}{\pi^2}\cosh(\pi t)\right)$, consequently $[\Re f_n] \in L_2([0;\tau])$. As $[\Re f_n](t) \to [\Re f]$ by the norm of the space $L_2\left(\mathbf{R}_+; \frac{4}{\pi^2}\cosh(\pi t)\right)$ and

$$\int_0^\tau |[\Re f_n](t) - [\Re f](t)|^2 dt < C\|[\Re f_n] - [\Re f]\|^2_{L_2\left(\mathbf{R}_+; \frac{4}{\pi^2}\cosh(\pi t)\right)}, \tag{6.100}$$

then $[\Re f_n] \to [\Re f]$ by the norm $L_2([0;\tau])$. Hence the Cauchy-Schwarz-Bunyakovskii inequality implies

$$\left|\int_0^\tau ([\Re f_n](t) - [\Re f](t))dt\right| \le \int_0^\tau |[\Re f_n](t) - [\Re f](t)|dt$$
$$\le \sqrt{\tau}||[\Re f_n] - [\Re f]||_{L_2([0;\tau])}. \tag{6.101}$$

Therefore,

$$\lim_{n\to\infty} \int_0^\tau [\Re f_n](t)dt = \int_0^\tau [\Re f](t)dt. \tag{6.102}$$

In the similar manner one can pass to the limit at the right-hand side of (6.98) preliminary derived an absolute convergence of the respective integral. Indeed, the function $f_n(x) \in L_2(\mathbf{R}_+)$ and moreover, we find that

$$\int_0^\infty |\Re K(\tau,y)f_n(y)|dy \le \left(\int_0^\infty |\Re K(\tau,y)|^2 dy\right)^{1/2} ||f_n||_{L_2(\mathbf{R}_+)}. \tag{6.103}$$

Let us show that for each $\tau > 0$ the function $\Re K(\tau,y) \in L_2(\mathbf{R}_+)$. Indeed, recall representation (6.84) and invoking with (6.99) obtain

$$\Re K(\tau,y) = C_\tau \int_0^\tau \cosh\left(\frac{\pi t}{2}\right) \Re K_{1/2+it}(y)dt$$
$$= \frac{C_\tau}{\sqrt{2}} \int_0^\tau dt \int_0^\infty \cos(tu)$$
$$\times [\cos(y\sinh u)\cosh(u/2) + \sin(y\sinh u)\sinh(u/2)]\, du, \tag{6.104}$$

where C_τ is a constant. However, according to the uniform convergence of integral (6.84) by τ one can perform the integration through in the right-hand side of (6.104) that gives

$$\Re K(\tau,y) = \frac{C_\tau}{\sqrt{2}} \int_0^\infty \frac{\sin(\tau u)}{u}$$
$$\times [\cos(y\sinh u)\cosh(u/2) + \sin(y\sinh u)\sinh(u/2)]\, du. \tag{6.105}$$

In the last integral make the replacement $v = \sinh u$ and consider the right-hand side of equality (6.105) as the sum of the cosine and the sine Fourier transforms with some coefficients that depend from y for each fixed $\tau \in \mathbf{R}_+$. For example, the first integral shall appear as

$$\int_0^\infty \cos(yv)\sqrt{(v^2+1)^{1/2}+1}\,\frac{\sin\left(\tau\log\left(v+(v^2+1)^{1/2}\right)\right)}{(v^2+1)^{1/2}\log\left(v+(v^2+1)^{1/2}\right)}dv. \tag{6.106}$$

The integrand here is a bounded function of $v > 0$ and equals $O(v^{-1/2}\log^{-1} v)$ at infinity. Consequently, it belongs to $L_2(\mathbf{R}_+)$ and as is known from the theory of Fourier integrals (see details in Titchmarsh [1]) the cosine Fourier transform related to (6.106) defines a L_2-function of $y \in \mathbf{R}_+$. In the same manner one can treat the second integral in (6.105), appealing to the property of the sine Fourier transform.

Thus, $\Re K(\tau, y) \in L_2(\mathbf{R}_+)$. Moreover, the limit relation $f_n \to f$ by the norm of $L_2(\mathbf{R}_+)$ and Cauchy-Schwarz-Bunyakovskii inequality imply that

$$\lim_{n\to\infty} \int_0^\infty \Re K(\tau, y) f_n(y)dy = \int_0^\infty \Re K(\tau, y) f(y)dy \tag{6.107}$$

and together with the limit in the left-hand side of equality (6.98) we obtain

$$\int_0^\tau [\Re f](t)dt = \int_0^\infty \Re K(\tau, y) f(y)dy. \tag{6.108}$$

Meanwhile, $[\Re f] \in L_2\left(\mathbf{R}_+; \frac{4}{\pi^2}\cosh(\pi t)\right)$ and it means $[\Re f] \in L_2((0, N])$. Therefore, we find that $[\Re f] \in L_1((0, N])$. Consequently, one can differentiate through in equality (6.108) and to deduce for almost all $\tau > 0$ the formula

$$[\Re f](\tau) = \frac{d}{d\tau} \int_0^\infty \Re K(\tau, y) f(y)dy. \tag{6.109}$$

Turning to the isometrical identity (6.83), being spread for all functions $f(x) \in L_2(\mathbf{R}_+)$ observe that the $\Re$-transform $[\Re f] \in L_2\left(\mathbf{R}_+; \frac{4}{\pi^2}\cosh(\pi\tau)\right)$ owing to the continuity of norms from the relation

$$||f_n||_{L_2(\mathbf{R}_+)} = ||[\Re f_n]||_{L_2\left(\mathbf{R}_+; \frac{4}{\pi^2}\cosh(\pi\tau)\right)}. \tag{6.110}$$

Hence as usually set in (6.82) $g(y) = 1,\ 0 < y \leq x,\ g(y) = 0, y > x$. This can be derived to the identity of kind

$$\int_0^x f(y)dy = \frac{4}{\pi^2} \int_0^\infty \cosh(\pi\tau) \int_0^x \Re K_{1/2+i\tau}(u)du[\Re f](\tau)d\tau. \tag{6.111}$$

However, recalling formula 1.12.1.2 from Prudnikov et al. [2] (see (2.71)), one can easily write the value of the inner integral at the right-hand side of (6.111), precisely

$$\int_0^x \Re K_{1/2+i\tau}(u)du$$

$$= \Re_{1/2+i\tau}\left[\frac{x^{1/2-i\tau}2^{-1/2+i\tau}\Gamma(1/2+i\tau)}{1/2-i\tau}\, {}_1F_2\left(\frac{1/2-i\tau}{2};\ 1/2-i\tau,\ \frac{5/2-i\tau}{2}; \frac{x^2}{4}\right)\right]$$
$$+\Re_{-1/2-i\tau}\left[\frac{x^{3/2-i\tau}2^{-3/2-i\tau}\Gamma(-1/2-i\tau)}{3/2-i\tau}\right.$$
$$\left.\times {}_1F_2\left(\frac{3/2-i\tau}{2};\ 3/2+i\tau,\ \frac{7/2-i\tau}{2}; \frac{x^2}{4}\right)\right]. \tag{6.112}$$

Hence, differentiating through in (6.111) by x and denoting the right-hand side of (6.112) by $S(x,\tau)$ we obtain the dual formula for the Lebedev-Skalskaya transform (6.109) as follows

$$f(x) = \frac{4}{\pi^2}\frac{d}{dx} \int_0^\infty \cosh(\pi\tau) S(x,\tau)[\Re f](\tau)d\tau. \tag{6.113}$$

In order to formulate the Plancherel theorem for the Lebedev-Skalskaya transform we may prove that formula (6.78) holds, precisely speaking our purpose is to establish that the Lebedev-Skalskaya transform $[\Re f]$ is the limit in mean by the norm of the space $L_2\left(\mathbf{R}_+; \frac{4}{\pi^2}\cosh(\pi\tau)\right)$ of the integral

$$\int_{1/N}^{N} \Re K_{1/2+i\tau}(y)f(y)dy$$

with an arbitrary L_2-function f. For this in the same manner as in Chapter 2 from equality

$$[\Re f_N](\tau) = \frac{d}{d\tau}\int_0^{\infty} \Re K(\tau,y)f_N(y)dy$$
$$= \frac{d}{d\tau}\int_{1/N}^{N} \Re K(\tau,y)f(y)dy \tag{6.114}$$

it follows that

$$[\Re f_N](\tau) = \int_{1/N}^{N} \Re K_{1/2+i\tau}(y)f(y)dy. \tag{6.115}$$

Consequently, invoking with (6.83) we arrive to the relation

$$||[\Re f] - [\Re f_N]||^2_{L_2\left(\mathbf{R}_+;\frac{4}{\pi^2}\cosh(\pi\tau)\right)} = ||f - f_N||^2_{L_2(\mathbf{R}_+)}$$
$$= \int_{y\notin[1/N,N]} |f(y)|^2 dy \to 0, \ N \to \infty, \tag{6.116}$$

which implies that $[\Re f_N] \to [\Re f]$ by the norm of the space $L_2\left(\mathbf{R}_+; \frac{4}{\pi^2}\cosh(\pi\tau)\right)$. Similarly we establish the convergence in mean of the sequence $\{f_N\}$ to f by the norm of L_2, if

$$f_N(x) = \frac{4}{\pi^2}\int_0^N \cosh(\pi\tau)\Re K_{1/2+i\tau}(x)[\Re f](\tau)d\tau. \tag{6.117}$$

Thus we proved

Theorem 6.8. *The operator of the Lebedev-Skalskaya transform given by formula* (6.109) *maps the space* $L_2(\mathbf{R}_+)$ *onto the space* $L_2\left(\mathbf{R}_+; \frac{4}{\pi^2}\cosh(\pi\tau)\right)$ *and almost everywhere on* $\mathbf{R}_+$ *formula* (6.113) *takes place. In addition, formulae* (6.78), (6.117) *are true as limits in mean by norms of* $L_2\left(\mathbf{R}_+; \frac{4}{\pi^2}\cosh(\pi\tau)\right)$ *and* $L_2(\mathbf{R}_+)$, *respectively.*

6.4 Convolution representations

As we saw in Chapter 4 it was established the series of miscellaneous results, concerning convolution operator (4.1) being related to the Kontorovich-Lebedev transform. Here one can succeed drawing a parallel with the respective representations for the Lebedev-Skalskaya transforms (6.74), (6.76). The key formulae in these cases are (6.23)-(6.24). Furthermore, we appeal to the Wiener ring L^{α} normed by (4.39), letting there $\alpha = 1/2$. As a result we reduce this space to the Lebesgue space of summable functions with a power-exponential weight.

Let us introduce the following convolution operator

$$(f * g)_{\Re}(x)$$

$$= \frac{1}{2\sqrt{2\pi}} \int_0^\infty \int_0^\infty \exp\left(-\frac{1}{2}\left(\frac{xy}{u} + \frac{yu}{x} + \frac{ux}{y}\right)\right)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{u}\right) f(u)g(y)dudy. \quad (6.118)$$

As is easy to see that (6.118) has the expression through the convolution of the Kontorovich-Lebedev transform (4.1) by means of the formula

$$(f * g)_{\Re}(x) = \frac{1}{\sqrt{2\pi}}\left[(f * g)(x) + x\left(f * \frac{g(y)}{y}\right)(x) + x\left(\frac{f(u)}{u} * g\right)(x)\right]. \quad (6.119)$$

In order to derive the factorization property related to convolution (6.118) and Parseval's type equality first deduce the integral representation for the kernel in double integral (6.118). For this call Theorem 6.4. Indeed, owing to Macdonald's type formula (6.23) and estimate (6.11) one can hold the limit passage almost for all $x > 0$ in the respective approximation operator (6.39) and obtain the identity

$$\exp\left(-\frac{1}{2}\left(\frac{xy}{u} + \frac{yu}{x} + \frac{ux}{y}\right)\right)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{u}\right)$$

$$= \frac{16}{\pi^2}\int_0^\infty \cosh(\pi\tau)\Re K_{1/2+i\tau}(x)\Re K_{1/2+i\tau}(y)\Re K_{1/2+i\tau}(u)d\tau. \quad (6.120)$$

Moreover, integral (6.120) is absolutely convergent according to the inequality

$$\int_0^\infty \cosh(\pi\tau)\left|\Re K_{1/2+i\tau}(x)\Re K_{1/2+i\tau}(y)\Re K_{1/2+i\tau}(u)\right| d\tau$$

$$\leq \frac{\pi}{2\cos\delta}\sqrt{\frac{\pi}{2yxu\cos\delta}}e^{-\cos\delta(x+y+u)}$$

$$\times \int_0^\infty e^{(\pi-3\delta)\tau}d\tau, \quad (6.121)$$

where $\pi/3 < \delta < \pi/2$.

As we noted above to construct a suitable Banach algebra for the introduced convolution (6.118) one can take functions from the ring $L^{1/2}$. More precisely, let us assume that an arbitrary function f belongs to the normed ring $L_1(\mathbf{R}_+; e^{-x}/\sqrt{x})$ with

$$||f||_{L_1(\mathbf{R}_+;e^{-x}/\sqrt{x})} = \int_0^\infty \frac{e^{-t}}{\sqrt{t}}|f(t)|dt < \infty. \quad (6.122)$$

Theorem 6.9. *Let $f(x)$, $g(x)$ be from the space $L_1(\mathbf{R}_+; e^{-x}/\sqrt{x})$. Then convolution* (6.118) *exists for almost all $x \in \mathbf{R}_+$ and belongs to $L_1(\mathbf{R}_+; e^{-x}/\sqrt{x})$. Furthermore,*

$$||(f * g)_{\Re}||_{L_1(\mathbf{R}_+;e^{-x}/\sqrt{x})} \leq ||f||_{L_1(\mathbf{R}_+;e^{-x}/\sqrt{x})}||g||_{L_1(\mathbf{R}_+;e^{-x}/\sqrt{x})}. \tag{6.123}$$

Proof. Since functions $f, g \in L_1(\mathbf{R}_+; e^{-x}/\sqrt{x})$, by definition of the norm (6.122) we obtain

$$||(f * g)_{\Re}||_{L_1(\mathbf{R}_+;e^{-x}/\sqrt{x})} = \int_0^\infty \frac{e^{-x}}{\sqrt{x}}|(f * g)_{\Re}(x)|dx. \tag{6.124}$$

Consequently, substituting the value of convolution (6.118) into the right-hand side of (6.124) one can estimate it as follows

$$||(f * g)_{\Re}||_{L_1(\mathbf{R}_+;e^{-x}/\sqrt{x})} \leq \int_0^\infty \frac{e^{-x}}{\sqrt{x}}dx$$

$$\times \frac{1}{2\sqrt{2\pi}} \int_0^\infty \int_0^\infty \exp\left(-\frac{1}{2}\left(\frac{xy}{u} + \frac{uy}{x} + \frac{xu}{y}\right)\right)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{u}\right)$$

$$\times |f(u)g(y)|dydu. \tag{6.125}$$

Thus according to conditions of this theorem one can change the order of integration in (6.125) and calculate the inner integral by formula (6.25). It immediately leads to the right-hand side of the desired inequality (6.123) as decomposed double integral in the product of two integrals (6.122) by y and u. The existence of the convolution (6.118) for almost all $x \in \mathbf{R}_+$ provided by the finiteness of norm (6.124). Theorem 6.9 is proved. •

The next theorem is a straightforward corollary of the Macdonald type formula (6.23) for the product of $\Re$-functions $\Re K_{1/2+i\tau}(x)\Re K_{1/2+i\tau}(y)$.

Theorem 6.10. *Let $f(x)$, $g(x) \in L_1(\mathbf{R}_+; e^{-x}/\sqrt{x})$. Then the Lebedev-Skalskaya transform* (6.43) *of convolution* (6.118) *$(f * g)_{\Re}(x)$ for functions $f(x)$ and $g(x)$ exists and equals to the product of the Lebedev-Skalskaya transforms for these functions, namely the factorization property*

$$[\Re(f * g)_{\Re}](\tau) = \sqrt{\frac{2}{\pi}}[\Re f](\tau)[\Re g](\tau), \tau \geq 0, \tag{6.126}$$

holds. In addition, if f, g being taken from a subspace $L_1(\mathbf{R}_+; e^{-\beta x}/\sqrt{x})$ with some fixed $0 < \beta < 1$, then the corresponding convolution representation of Parseval's type yields the form

$$(f * g)_{\Re}(x) = \left(\frac{2}{\pi}\right)^{5/2} \int_0^\infty \cosh(\pi\tau)\Re K_{1/2+i\tau}(x)[\Re f](\tau)[\Re g](\tau)d\tau \tag{6.127}$$

as well as the uniform estimate is true

$$|(f*g)_{\Re}(x)| \le C_\beta \frac{e^{-\beta x}}{\sqrt{x}} ||f||_{L_1(\mathbf{R}_+;e^{-\beta x}/\sqrt{x})} ||g||_{L_1(\mathbf{R}_+;e^{-\beta x}/\sqrt{x})},\ x>0. \tag{6.128}$$

Proof. Indeed, according to the previous theorem one can conclude that the Lebedev-Skalskaya transform (6.43) of convolution (6.118) exists. Therefore, apply it through (6.118) after interchanging the order of integration by Fubini's theorem invoke with identity (6.23) and obtain the chain of equalities

$$\begin{aligned}
[\Re(f*g)_{\Re}](\tau) &= \int_0^\infty \Re K_{1/2+i\tau}(x)(f*g)_{\Re}(x)dx \\
&= \frac{1}{2\sqrt{2\pi}} \int_0^\infty \Re K_{1/2+i\tau}(x)dx \int_0^\infty \int_0^\infty f(u)g(y) \\
&\times \exp\left(-\frac{1}{2}\left(\frac{xy}{u}+\frac{yu}{x}+\frac{ux}{y}\right)\right)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{u}\right) dudy \\
&= \sqrt{\frac{2}{\pi}} \int_0^\infty \Re K_{1/2+i\tau}(u)f(u)du \int_0^\infty \Re K_{1/2+i\tau}(y)g(y)dy \\
&= \sqrt{\frac{2}{\pi}} [\Re f](\tau)[\Re g](\tau).
\end{aligned} \tag{6.129}$$

Thus we arrived to (6.126). To derive formula (6.127) and estimate (6.128) turn to identity (6.120) and integrate it through by y and u after multiplication both of sides on the product $\left(2\sqrt{2\pi}\right)^{-1} f(u)g(y)$. The changing of the order of integration can be performed by using inequality (6.121), letting there $\beta = \cos\delta,\ \pi/3 < \delta < \pi/2$ and conditions of this theorem. As a result apply the Fubini theorem and come to the desired identity (6.127). Inequality (6.128) can be immediately obtained from (6.121), namely

$$\begin{aligned}
|(f*g)_{\Re}(x)| &\le C_\beta \frac{e^{-\beta x}}{\sqrt{x}} \\
\times \int_0^\infty \frac{e^{-\beta u}}{\sqrt{u}} |f(u)|du &\int_0^\infty \frac{e^{-\beta y}}{\sqrt{y}} |g(y)|dy.
\end{aligned} \tag{6.130}$$

This ends the proof of Theorem 6.10. •

Remark 6.2. Note here that in the same manner the algebraic properties of commutativity (4.4), associativity (4.56) and distributivity (4.57) for the $\Re$-convolution (6.118) can be derived as for the Kontorovich-Lebedev convolution (4.1).

Now we begin to consider the convolution Hilbert space for convolution (6.118). Theorem 6.10 gives us that if $f,g \in L^\beta \equiv L_1(\mathbf{R}_+; e^{-\beta x}/\sqrt{x}),\ 0<\beta<1$, then owing to estimates (6.128), (6.130) it follows that $(f*g)_{\Re} \in L^\beta$ and furthermore,

$$||(f*g)_{\Re}||_{L^\beta} \le \hat{C}_\beta ||f||_{L^\beta} ||g||_{L^\beta}, \tag{6.131}$$

where $\hat{C}_\beta$ is a positive constant being depended from β. Call equality (6.127) for two complex-valued functions $f, \bar{g} \in L^\beta$ and an arbitrary positive function $\omega(x)$ being defined on $\mathbf{R}_+$ such that $\omega \in L^\beta$. Hence as is evident, by virtue of (6.128) the following integral

$$\int_0^\infty (f * \bar{g})_{\Re}(x)\omega(x)dx$$

is an absolutely convergent. On the other hand, multiplying through the respective equality (6.127) by $\omega(x)$ and integrating through by $x \in \mathbf{R}_+$ we find that

$$\int_0^\infty (f * \bar{g})_{\Re}(x)\omega(x)dx$$

$$= \left(\frac{2}{\pi}\right)^{5/2} \int_0^\infty \omega(x)dx \int_0^\infty \cosh(\pi\tau)\Re K_{1/2+i\tau}(x)[\Re f](\tau)[\Re\bar{g}](\tau)d\tau. \tag{6.132}$$

The above conditions on functions f, g and the function $\omega(x)$ enable to change the order of integration in view of the absolutely convergent iterated integral (6.132). Therefore, one can write the equality

$$\int_0^\infty (f * \bar{g})_{\Re}(x)\omega(x)dx$$

$$= \left(\frac{2}{\pi}\right)^{5/2} \int_0^\infty \cosh(\pi\tau)\varrho(\tau)[\Re f](\tau)[\Re\bar{g}](\tau)d\tau, \tag{6.133}$$

where the weighted function $\varrho(\tau)$, $\tau \in \mathbf{R}_+$ supposed to be a positive one and defined by the integral

$$\varrho(\tau) = \int_0^\infty \Re K_{1/2+i\tau}(x)\omega(x)dx. \tag{6.134}$$

Now to require the left-hand side of (6.133) to be an inner product and denote it by

$$\int_0^\infty (f * \bar{g})_{\Re}(x)\omega(x)dx = \langle f, g\rangle, \tag{6.135}$$

it remains to show that the identity $\langle f, f\rangle \equiv 0$ is valid if and only if $f = 0$ almost everywhere on $\mathbf{R}_+$. Indeed, this identity leads to the following one

$$\int_0^\infty \cosh(\pi\tau)\varrho(\tau)|[\Re f](\tau)|^2 d\tau \equiv 0, \tag{6.136}$$

and consequently, owing to the assumed positiveness of the weighted function ϱ it means that $[\Re f](\tau) \equiv 0$, because of the continuity of the Lebedev-Skalskaya transform (6.43), when $f \in L^\beta$. Meanwhile, according to integral representation (6.5) $\Re$-transform (6.43) can be written as a composition of the Laplace and the cosine Fourier transforms with multiplier in the same manner as in Theorem 4.12, namely

$$[\Re f](\tau) = \sqrt{\frac{\pi}{2}} \left[F_c \cosh\left(\frac{u}{2}\right)[Lf](\cosh u)\right](\tau). \tag{6.137}$$

This immediately implies that $[Lf](\cosh u) \equiv 0$. Finally, appealing to Theorem 4.12 obtain the desired result, that $f = 0$ almost for all $x \in \mathbf{R}_+$.

Thus it is straightforward that $\langle f,g\rangle$ defined by (6.135) possesses by all properties of the inner product. With this inner product the set of functions L^β becomes the pre-Hilbert space. Usual completion procedure of it brings to the convolution Hilbert space denoted by S_ϱ. So for any elements $f \in S_\varrho$, $g \in S_\varrho$ the inner product $\langle f,g\rangle$ is defined as well as the norm $||f||_S = \sqrt{\langle f,f\rangle}$. If $f \in L^\beta$, $g \in L^\beta$, then invoking with (6.118) we find that

$$\langle f,g\rangle = \int_0^\infty (f * \bar{g})_{\mathfrak{R}}(x)\omega(x)dx$$

$$= \frac{1}{2\sqrt{2\pi}} \int_0^\infty \omega(x)dx \int_0^\infty \int_0^\infty \exp\left(-\frac{1}{2}\left[\frac{xu}{y} + \frac{xy}{u} + \frac{yu}{x}\right]\right)$$

$$\times \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{u}\right) f(u)\overline{g(y)}dudy$$

$$= \int_0^\infty \int_0^\infty S_\omega^{\mathfrak{R}}(u,y) f(u)\overline{g(y)}dudy, \tag{6.138}$$

where we mean by

$$S_\omega^{\mathfrak{R}}(u,y) = \frac{1}{2\sqrt{2\pi}} \int_0^\infty \omega(x) \exp\left(-\frac{1}{2}\left[\frac{xu}{y} + \frac{xy}{u} + \frac{yu}{x}\right]\right) \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{u}\right) dx. \tag{6.139}$$

In fact, the positiveness of the inner product $\langle f,f\rangle$ in the form

$$\langle f,f\rangle = \int_0^\infty \int_0^\infty S_\omega^{\mathfrak{R}}(u,y) f(u)\overline{f(y)}dudy, \tag{6.140}$$

is provided, for instance, by the right-hand side of (6.133) setting there $f = g$. Moreover, if $f(x)$ satisfies the condition

$$\int_0^\infty \int_0^\infty S_\omega^{\mathfrak{R}}(u,y)|f(u)f(y)|dudy < \infty, \tag{6.141}$$

then $||f||_{S_\varrho} < \infty$ and $f \in S_\varrho \supset L^\beta$. Furthermore, if $f(x)$ and $g(x)$ satisfy (6.141), then similar to (4.105) we find that

$$|\langle f,g\rangle| \le ||f||_{S_\varrho}||g||_{S_\varrho}, \tag{6.142}$$

and it implies that the integral

$$\int_0^\infty \int_0^\infty S_\omega^{\mathfrak{R}}(u,y)|f(u)g(y)|dudy \tag{6.143}$$

is convergent being satisfied to equality (6.138).

As in Chapter 4 denote by $H_\varrho \equiv L_2(\mathbf{R}_+; (2/\pi)^{5/2}\cosh(\pi\tau)\varrho(\tau))$ the corresponding Hilbert space of images for the Lebedev-Skalskaya transforms $h(\tau)$ normed by

$$||h||_{H_\varrho}^2 = \left(\frac{2}{\pi}\right)^{5/2} \int_0^\infty \cosh(\pi\tau)\varrho(\tau)|h(\tau)|^2 d\tau, \tag{6.144}$$

where $\varrho(\tau)$ defined by formula (6.134). In addition, the inner product of two functions $\varphi, \psi \in H_\varrho$ can be derived as usually

$$(\varphi, \psi) = \left(\frac{2}{\pi}\right)^{5/2} \int_0^\infty \cosh(\pi\tau)\varrho(\tau)\varphi(\tau)\overline{\psi(\tau)}d\tau. \tag{6.145}$$

So equality (6.133) shows that the operator of the Lebedev-Skalskaya transform (6.43) maps the space L^β into H_ϱ and moreover,

$$||[\Re f]||^2_{H_\varrho} = \left(\frac{2}{\pi}\right)^{5/2} \int_0^\infty \cosh(\pi\tau)\varrho(\tau)|[\Re f](\tau)|^2 d\tau$$

$$= \int_0^\infty (f * \bar{f})(x)\omega(x)dx = ||f||^2_{S_\varrho}. \tag{6.146}$$

Consequently, due to the Banach theorem one can extend by the continuity the Lebedev-Skalskaya operator for all $f \in S_\varrho$. Thus operator (6.43) is defined for all $f \in S_\varrho$, its range $LS(S_\varrho)$ belongs to H_ϱ and for any $f \in S_\varrho$ we have

$$||f||_{S_\varrho} = ||[\Re f]||_{H_\varrho}, \ [\Re f](\tau) = 0, \text{ iff } f = 0. \tag{6.147}$$

Hence this implies that there exists the inverse bounded operator $[\Re^{-1}h]$. Appealing to Theorem 6.10 conclude that if $f,\ g \in L^\beta$ then $(f * g)_\Re(x) \in L^\beta$ and furthermore, equality (6.126) is valid. Consequently, we find that

$$(f * g)_\Re(x) = \sqrt{\frac{2}{\pi}} \left[\Re^{-1}\ [\Re f][\Re g]\right](x). \tag{6.148}$$

It implies, in turn, that if for two elements of the convolution Hilbert space S_ϱ f, g the product $[\Re f](\tau)[\Re g](\tau) = \varphi(\tau)\psi(\tau) \in LS(S_\varrho)$, then the element $(2/\pi)^{1/2}\,[\Re^{-1}\ [\varphi\psi]]$ is called the generalized $\Re$-convolution of elements f, g.

In the same manner as in Chapter 4 prove that for any $f \in S_\varrho$ and $g \in L^\beta$ convolution (6.118) exists and moreover,

$$||(f * g)_\Re||_{S_\varrho} \le \sup_{\tau > 0} |\psi(\tau)|\ ||f||_{S_\varrho}, \tag{6.149}$$

where we mean

$$\psi(\tau) \equiv [\Re g](\tau) = \int_0^\infty \Re K_{1/2+i\tau}(y)g(y)dy. \tag{6.150}$$

Indeed, since $g \in L^\beta$ then by virtue (6.11) with $\beta = \cos\delta$ we find

$$|\psi(\tau)| \le \sqrt{\frac{\pi}{2\beta}} e^{-\beta\tau}||g||_{L^\beta}. \tag{6.151}$$

Therefore, as is evident $\sup_{\tau>0}|\psi(\tau)| = M < \infty$ and consequently, $\varphi(\tau)\psi(\tau) \in H_\varrho$, $\varphi(\tau) \equiv [\Re f](\tau)$.

It remains to show that $\varphi(\tau)\psi(\tau) \in LS(S_\varrho)$. The proof is straightforward by the choice of some sequence $f_n(x) \in L^\beta$ such that $||f - f_n||_{S_\varrho}$ tends to zero as n tends to infinity. Hence according to (6.128) $h_n(x) = (f_n * g)_{\Re}(x) \in L^\beta$. Denoting by $\varphi_n(\tau) = [\Re f_n](\tau)$ and invoking with (6.144) we have

$$||h_n - h_m|| = ||(f_n - f_m) * g||_{S_\varrho} = ||[\Re[f_n - f_m]][\Re g]||_{H_\varrho}$$

$$= ||(\varphi_n - \varphi_m)\psi||_{H_\varrho} \le M||\varphi_n - \varphi_m||_{H_\varrho} = M||f_n - f_m||_{S_\varrho}. \tag{6.152}$$

Hence conclude that the sequence h_n is convergent at the Hilbert space S_ϱ. Put the corresponding limit by h. Then it follows that

$$[\Re h](\tau) = \sqrt{\frac{2}{\pi}}[\Re f](\tau)[\Re g](\tau) = \varphi(\tau)\psi(\tau). \tag{6.153}$$

Thus we obtained the desired result that the product $\varphi(\tau)\psi(\tau)$ belongs to $LS(S_\varrho)$.

We are ready now to establish the analog of Theorem 4.19 being dealt with the inversion formula for the Lebedev-Skalskaya transform (6.43) in the convolution Hilbert space S_ϱ.

Theorem 6.11. *Let the weighted function $\omega(x)$ be from the space L^β. Then for functions $f \in S_\varrho$ for all $x > 0$ the following inversion formula of the Lebedev-Skalskaya transform is true*

$$(f * \omega)_{\Re}(x) = \left(\frac{2}{\pi}\right)^{5/2} \frac{d}{dx}\int_0^\infty \cosh(\pi\tau)\varrho(\tau)S(x,\tau)[\Re f](\tau)d\tau, \tag{6.154}$$

where the kernel $S(x,\tau)$ defined by equality (6.112). In particular, if $f \in L^\beta \subset S_\varrho$, then

$$(f * \omega)_{\Re}(x) = \left(\frac{2}{\pi}\right)^{5/2} \int_0^\infty \cosh(\pi\tau)\varrho(\tau)\Re K_{1/2+i\tau}(x)[\Re f](\tau)d\tau. \tag{6.155}$$

Proof. Setting in (6.133) $g(y) = 1,\ 0 < y \le x;\ g(y) = 0,\ y > x$ and invoking with identity (6.112) we derive the formula

$$\int_0^x \int_0^\infty S_\omega^{\Re}(u,y)f(u)dudy$$

$$= \left(\frac{2}{\pi}\right)^{5/2} \int_0^\infty \cosh(\pi\tau)\varrho(\tau)S(x,\tau)[\Re f](\tau)d\tau. \tag{6.156}$$

Hence the desired results follow after differentiation through by x taking into account definition (6.118) of the convolution. In addition, we note that if $f \in L^\beta$ that it gives the possibility to perform the differentiation under the sign of the integral in the right-hand side of (6.156). This completes the proof of Theorem 6.11. •

The last theorem of this chapter shows coincidence of the range $LS(S_\varrho)$ with the Hilbert space H_ϱ.

Theorem 6.12. *The range of the Lebedev-Skalskaya transform $SL(S_\varrho)$ coincides with the Hilbert space H_ϱ.*

Proof. In fact, it means that there no exists in H_ϱ any element except zero that is orthogonal to $LS(S_\varrho)$. Hence, by formula (6.145) we have, that $(\varphi_0, [\Re g]) = 0$ for arbitrary $g \in S_\varrho$. In particular, take the function g as $g(y) = 1,\ 0 < y \le x;\ g(y) = 0,\ y > x$. Meanwhile, the equality

$$\left(\varphi_0, \int_0^x \Re K_{1/2+i\tau}(y)dy\right) = \left(\frac{2}{\pi}\right)^{5/2} \int_0^\infty \cosh(\pi\tau)\varrho(\tau)\varphi_0(\tau) \times \int_0^x \Re K_{i\tau}(y)dyd\tau = 0 \tag{6.157}$$

after differentiation by x yields

$$\int_0^\infty \cosh(\pi\tau)\varrho(\tau)\varphi_0(\tau)\Re K_{1/2+i\tau}(x)d\tau = 0 \tag{6.158}$$

for all $x > 0$. It is possible owing to absolute and uniform convergence of the integral (6.18) in view of estimate

$$\int_0^\infty \cosh(\pi\tau)\varrho(\tau)|\varphi_0(\tau)\Re K_{1/2+i\tau}(x)|d\tau$$
$$\le C\frac{e^{-x\beta}}{\sqrt{x}}||\varphi_0||_{H_\varrho}\left(\int_0^\infty \cosh(\pi\tau)\varrho(\tau)e^{-2\beta\tau}d\tau\right)^{1/2} < \infty. \tag{6.159}$$

Consequently, the left hand-side of (6.158) is a L_1-function on $\mathbf{R}_+$. Further, as is known by using inversion formula (6.75) for the Lebedev-Skalskaya transform (6.74) and index integral 2.16.54.3 in Prudnikov et al. [2], one can deduce the following identity

$$\sqrt{\frac{2}{\pi}}\int_x^\infty \frac{e^{x-t}}{\sqrt{x-t}}\Re K_{1/2+i\tau}(t)dt = K_{i\tau}(x),\ x > 0. \tag{6.160}$$

Then multiply through in equality (6.158) by the integrand of (6.160) and perform after the integration with changing the order by the Fubini theorem in view of estimate (6.159). As a result we arrive to the equation like (4.128), namely

$$\int_0^\infty \cosh(\pi\tau)\varrho(\tau)\varphi_0(\tau)K_{i\tau}(x)d\tau = 0. \tag{6.161}$$

Applying, in turn, through in (6.161) the cosine Fourier transform (1.197) , change again the order of integration and by formula 2.16.14.1 in Prudnikov et al. [2] we obtain that

$$\int_0^\infty \frac{\cosh(\pi\tau)}{\cosh(\pi\tau/2)}\varrho(\tau)\varphi_0(\tau)\cos\left(\tau\log(x+\sqrt{x^2+1})\right)d\tau \equiv 0. \tag{6.162}$$

In the same manner as in Chapter 4 conclude that the integrand in (6.162) belongs to the space $L_1(\mathbf{R}_+)$ by τ. Thus we led that $\varphi_0(\tau) = 0$ almost everywhere. Theorem 6.12 is proved. •

Chapter 7

Index Transforms with Hypergeometric Functions in The Kernel

This final chapter completes the presentation of the theory of the index transforms by various examples of operators being depended upon the parameters of special functions of hypergeometric type. Here we shall study several index transforms that involve as the corresponding kernel the Gauss hypergeometric function (1.47), the Whittaker function (1.131), the Appel F_3-function (1.140) and certain combinations of the Bessel functions which were introduced in Chapter 1. As is known these special functions are particular cases of the general Meijer's G-function (1.59) and Fox's H-function (1.63). In this volume (see lines (1.107)-(1.140)) we listed their expressions as well as other formulae to show the important connection between special functions of hypergeometric type and the theory of the Mellin-Barnes integrals. Consequently, we possess now by the common point of view to investigate the respective index transforms. We shall be based on the Mellin-Barnes type integral representations in L_p-spaces and the compositions through the Kontorovich-Lebedev transform as well as the Mellin convolution type operators. Note, that the general index transforms with arbitrary kernels have been considered in Chapters 2 and 5. Nevertheless, we draw here special attention and single out the particular cases mentioned above by virtue of the exceptional properties of the kernels and the possibility of their explicit inversions. Furthermore, we spread this approach to introduce the essentially multidimensional Kontorovich-Lebedev transform by means of the composition of the multidimensional Fourier transform and special modification of the multidimensional Laplace transform. Concerning this modification of the Laplace transform we refer the reader to more detailed information on this matter in the book by Brychkov et al. [1, 1992] (see also Nguyen Thanh Hai et al. [1], Vu Kim Tuan [3], [6], Vu Kim Tuan and Nguyen Thanh Hai [1]).

Historically, the index transform depending upon a parameter of Gauss's function first appeared in Olevskii [1] and it was named as the Olevskii transform or the index ${}_2F_1$- transform. Later it was investigated by the author for instance, in Yakubovich [1], [4], Yakubovich et al. [1, 1987]. Let us mention that recently the cycle of papers of Hayek et al. [1, 1990,1992], Hayek, N. and Gonzalez, B.J. [2]-[3], Hayek, C.N. and Gonzalez, R.B. [1]-[2] has been devoted to the consideration of the index ${}_2F_1$-transform including its distributional analog. Concerning index transforms involving different hypergeometric functions as the kernels the reader can find in Lebedev [7]-[8], Marichev [1], Wimp [1], Brychkov et al. [1, 1986], Vu Kim Tuan et al. [1], Yakubovich and Luchko [2], Yakubovich et al. [1, 1987,1994].

Our key purpose in this chapter is to apply the L_p-theory of the Kontorovich-Lebedev transform and the Mellin transform technique exhibited above for investigation the L_p-properties of the index transforms with hypergeometric functions as the kernels and obtaining their explicit inversions.

7.1 Index transforms of the Olevskii type

We may introduce here first *the Olevskii transform* being contained the special case of the Gauss function as the kernel with symmetric parameters. However, let us mention beforehand that we already investigated certain of its particular cases, when we considered the generalized Mehler-Fock transform (3.93). Consequently, taking an arbitrary function f from suitable space of functions being defined below we introduce the following index operator

$$
{}_2F_1^{i\tau}[f] = 2^{\alpha-2}\frac{\Gamma((\alpha+\mu+i\tau)/2)\Gamma((\alpha+\mu-i\tau)/2)}{\Gamma(\mu+1)}
$$

$$
\times\int_0^\infty v^{\alpha+\mu-1}{}_2F_1\left(\frac{\alpha+\mu+i\tau}{2},\frac{\alpha+\mu-i\tau}{2};\mu+1;-v^2\right)f(v)dv. \qquad (7.1)
$$

As it is easily seen similar to representation (3.92) one can derive the kernel of the Olevskii transform by integral (1.101). Therefore, we assume that arbitrary parameters α,μ satisfy the condition $\Re(\alpha+\mu)>0$. By simple interchanges the left-hand side of identity (1.101) can be reduced to the index kernel (5.69) with the corresponding function θ as the Bessel function with a power multiplier. Hence one can apply the general results from Chapter 5. Nevertheless, we shall study the Olevskii transform (7.1) independently and shall be based on the known asymptotic properties of the Bessel functions. Thus observe that integral (1.101) gives us the possibility to estimate uniformly by $(\tau,x)\in\mathbf{R}_+\times\mathbf{R}_+$ the Gauss hypergeometric function in (7.1) by using our key inequality (1.100). Namely, denoting by ${}_2F_1^{i\tau}(x)$ the kernel of (7.1)

$$
{}_2F_1^{i\tau}(x) = 2^{\alpha-2}\frac{\Gamma((\alpha+\mu+i\tau)/2)\Gamma((\alpha+\mu-i\tau)/2)}{\Gamma(\mu+1)}
$$

$$
\times x^{\mu}{}_2F_1\left(\frac{\alpha+\mu+i\tau}{2},\frac{\alpha+\mu-i\tau}{2};\mu+1;-x^2\right), \qquad (7.2)
$$

rewrite operator (7.1) as follows

$$ {}_2F_1^{i\tau}[f] = \int_0^\infty v^{\alpha-1}{}_2F_1^{i\tau}(v)f(v)dv \tag{7.3} $$

and we have the uniform estimate of kernel (7.2)

$$ \left|{}_2F_1^{i\tau}(x)\right| \le e^{-\delta\tau}\int_0^\infty y^{\Re\alpha-1}|J_\mu(xy)|K_0(y\cos\delta)dy, \tag{7.4} $$

where as usually $\delta \in [0,\pi/2)$. To continue inequality (7.4) use the same arguments as in Chapter 3 (see (3.96)) and finally we obtain

$$ \left|{}_2F_1^{i\tau}(x)\right| \le Ce^{-\delta\tau}x^{\Re\mu}\int_0^\infty y^{\Re(\alpha+\mu)-1}K_0(y\cos\delta)dy $$

$$ = C_{\alpha,\delta}x^{\Re\mu}e^{-\delta\tau}, (\tau,x) \in \mathbf{R}_+ \times \mathbf{R}_+, \tag{7.5} $$

provided that $\Re(\alpha+\mu) > 0, \Re\mu \ge -1/2$ and $C_{\alpha,\delta}$ is some positive constant. Further, applying formula (1.86) of the asymptotic behavior of Gauss's function observe that in our case for each $\tau > 0$ kernel (7.2) is $O(x^{-\Re\alpha}), x \to \infty$. Moreover, it is easily verified by means of equality (1.51) that ${}_2F_1^{i\tau}(x) = O(x^{\Re\mu}), x \to 0+$. Consequently, in a similar manner to (3.97)-(3.98) the Olevskii transform (7.3) of an arbitrary function $f \in L_{\nu,p}(\mathbf{R}_+)$ can be estimated as follows

$$ \left|{}_2F_1^{i\tau}[f]\right| \le Ce^{-\delta\tau}||f||_{\nu,p} $$

$$ \times \left[\left(\int_0^1 v^{(\Re(\alpha+\mu)-\nu)q-1}dv\right)^{1/q} + \left(\int_1^\infty v^{-\nu q-1}dv\right)^{1/q}\right]. \tag{7.6} $$

Clearly, that integrals in the right-hand side of (7.6) are convergent when $0 < \nu < \Re(\alpha+\mu)$. In addition, it gives us immediately the analog of Theorem 3.6 for the Olevskii transform (7.3).

Theorem 7.1. *The Olevskii transform* (7.3) *is a bounded operator as a mapping from the space* $L_{\nu,p}(\mathbf{R}_+)$ *with* $p \ge 1, 0 < \nu < \Re(\alpha+\mu), \Re\mu \ge -1/2$ *into the space* $L_r(\mathbf{R}_+), r \ge 1$. *In addition, the Olevskii transform can be factorized via a composition of the Kontorovich-Lebedev transform* (2.1) *and the Hankel transform* (1.225) *with power multiplier applied in the order*

$$ {}_2F_1^{i\tau}[f] = K_{i\tau}\left[x^{\alpha-3/2}\left[J_\mu\{x^{\alpha-3/2}f(x)\}\right]\right], \tau \ge 0. \tag{7.7} $$

Proof. Indeed, taking into account definitions of the Kontorovich-Lebedev and the Hankel transforms the right-hand side of equality (7.7) becomes the iterated integral. It equals the value of the Olevskii transform (7.3) on the function $f \in L_{\nu,p}(\mathbf{R}_+)$ because of representation (1.101) is true, and one can perform the changing of the order of integration by Fubini's theorem owing to the above estimate (7.6). •

It is natural now to invert composition (7.7) and to give the inversion formula of the Olevskii transform which as is evident generalizes the respective formula (3.100) for the Mehler-Fock transform (3.93). The following statement is valid.

Theorem 7.2. *Let $f \in L_{\nu,p}(\mathbf{R}_+)$, where $1 < p \leq 2$ and $\max(0, p^{-1} + \Re\alpha - 3/2) < \nu < \min(1, \Re(\alpha + \mu))$, $\Re\mu \geq -1/2, \Re\alpha < \Re\mu + 2$. Then the inversion formula for the Olevskii transform*

$$f(x) = \frac{x^{2(1-\alpha)+\mu}}{\pi^2\Gamma(\mu+1)} \text{l.i.m.}_{\varepsilon\to 0+} 2^{\varepsilon-\alpha+1} \int_0^\infty \tau \sinh((\pi-\varepsilon)\tau)$$

$$\times \Gamma\left(1 + \frac{\varepsilon - \alpha + \mu + i\tau}{2}\right) \Gamma\left(1 + \frac{\varepsilon - \alpha + \mu - i\tau}{2}\right)$$

$$\times {}_2F_1\left(1 + \frac{\varepsilon - \alpha + \mu + i\tau}{2}, 1 + \frac{\varepsilon - \alpha + \mu - i\tau}{2}; 1 + \mu; -x^2\right) {}_2F_1^{i\tau}[f] d\tau \tag{7.8}$$

is valid.

Proof. Clearly, that to establish formula (7.8) first one may examine composition (7.7) on the matter to satisfy the inversion Theorems 1.21 and 2.3 of the Kontorovich-Lebedev and the Hankel transforms, respectively. For this observe that since $f \in L_{\nu,p}(\mathbf{R}_+)$ then the product $x^{\alpha-3/2} f(x)$ belongs to the space $L_{\nu-\alpha+3/2,p}(\mathbf{R}_+)$. Hence the conditions of the present theorem allows us to conclude that $[J_\mu\{x^{\alpha-3/2}f(x)\}] \in L_{\alpha-\nu-1/2,p}(\mathbf{R}_+), 1 < p \leq 2$. From composition (7.7) and Theorem 2.3 it follows that ${}_2F_1^{i\tau}[f] \in L_{1-\nu,p}(\mathbf{R}_+)$ with $\nu \in (0,1)$ according to the condition on the parameter ν of the theorem. Thus inverting the Kontorovich-Lebedev transform by formula (2.19) we find that

$$\left[J_\mu\{x^{\alpha-3/2}f(x)\}\right] = \frac{2}{\pi^2} \text{l.i.m.}_{\varepsilon\to 0+} \int_0^\infty \tau \sinh((\pi-\varepsilon)\tau) x^{\varepsilon-\alpha+1/2} K_{i\tau}(x) {}_2F_1^{i\tau}[f] d\tau, \tag{7.9}$$

where the limit is meant by the norm of $L_{\alpha-\nu-1/2,p}$-space. The Hankel transform, in turn, can be inverted by Theorem 1.21 and moreover, one can carry out the limit sign owing to its boundedness as the operator from the space $L_{\nu-\alpha+3/2,p}(\mathbf{R}_+)$ into the space $L_{\alpha-\nu-1/2,p}(\mathbf{R}_+)$. Therefore it gives us the following iterated integral

$$x^{\alpha-2} f(x) = \frac{2}{\pi^2} \text{l.i.m.}_{\varepsilon\to 0+} \int_0^\infty y^{\varepsilon-\alpha+1} J_\mu(xy) dy$$

$$\times \int_0^\infty \tau \sinh((\pi-\varepsilon)\tau) K_{i\tau}(y) {}_2F_1^{i\tau}[f] d\tau. \tag{7.10}$$

The above estimate (7.6) and inequality (1.100) enable us to change the order of integration within integral (7.10). Appealing to equality (1.101) under condition $\Re\alpha < \Re\mu + 2$ calculate the inner integral by y and arrive finally to the inversion formula (7.8) for the Olevskii transform (7.3). This completes the proof of Theorem 7.2. •

Now let us consider another example of the index transform by the second index of the Whittaker function (1.131). This transform was deduced in Wimp [1] as a particular case of the general expansion (1.237). Afterwards it has been studied by the author mainly its composition structure and connection with the Kontorovich-Lebedev transform. Let us mention the paper of Virchenko and Gamaleya [1] that contains the inversion L_2-theorem for Wimp's transform with the Whittaker function. As we saw in Chapter 1 owing to Slater's theorem G-function (1.131) can be expressed by means of formula (1.171). Furthermore, its asymptotic expansion, which is equivalent to the asymptotic formula by index of the Whittaker function is given by relation (1.172). Here we appeal in our discussions to the Mellin-Barnes integral related to Meijer's G-function in (1.131) to reduce it by the Mellin transform technique demonstrated above and to apply for the composition representation.

Thus we introduce the index transform with the Whittaker function as the kernel by the following integral

$$W_{\varrho,i\tau}[f] = \int_0^\infty W_{\varrho,i\tau}\left(\frac{1}{v^2}\right) e^{-(2v^2)^{-1}} f(v)dv, \tag{7.11}$$

where $\tau \geq 0, \varrho$ is some fixed complex parameter and an arbitrary function f possesses by the properties of $L_{\nu,p}$-functions. We start from the correspondence (1.131). Making use the reflection formula (1.61) for the respective G-function and simple interchange of variable in the Mellin-Barnes integral the kernel of the index transform (7.11) can be written in the form

$$W_{\varrho,i\tau}\left(\frac{1}{x^2}\right) e^{-(2x^2)^{-1}} = \frac{1}{4\pi i}\int_{\nu-i\infty}^{\nu+i\infty} \Gamma\left(\frac{1-s}{2}+i\tau\right)\Gamma\left(\frac{1-s}{2}-i\tau\right)$$

$$\times \frac{x^{-s}}{\Gamma(1-\varrho-s/2)}ds, \tag{7.12}$$

where we assume that $\nu < \min(1, 2(1-\Re\varrho))$. We need this assumption to provide the positiveness of the real parts of gamma-functions in the gamma-ratio of integrand (7.12). So, if $f \in L_{1-\nu,p}(\mathbf{R}_+), 1 < p \leq 2$, then in accordance with Theorem 1.17 we obtain the representation by the Mellin-Parseval equality as

$$W_{\varrho,i\tau}[f] = \frac{1}{4\pi i}\int_{\nu-i\infty}^{\nu+i\infty} \Gamma\left(\frac{1-s}{2}+i\tau\right)\Gamma\left(\frac{1-s}{2}-i\tau\right)\frac{f^*(1-s)}{\Gamma(1-\varrho-s/2)}ds. \tag{7.13}$$

Meanwhile, recall identity (2.125), which immediately gives another representation of the index transform (7.11) through the Kontorovich-Lebedev transform, namely

$$W_{\varrho,i\tau}[f] = \int_0^\infty K_{2i\tau}(y)f_1(y)dy, \tag{7.14}$$

where the function f_1 is defined by formula

$$f_1(x) = \frac{1}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty} \frac{2^s f^*(1-s)}{\Gamma(1-\varrho-s/2)}x^{-s}ds. \tag{7.15}$$

In addition, let us assume that integral (7.15) is absolutely convergent. Then to invert the index transform with the Whittaker function invoke with identity 2.19.28.7 in Prudnikov et al. [3], which contains the index integral from the product of the Macdonald and the Whittaker function. More precisely, we have

$$\int_0^\infty \tau \sinh(2\pi\tau)\Gamma\left(\frac{1}{2}-\varrho+i\tau\right)\Gamma\left(\frac{1}{2}-\varrho-i\tau\right)K_{2i\tau}(y)W_{\varrho,i\tau}(x)d\tau$$

$$= \pi^2 2^{2(\varrho-1)}y^{1-2\varrho}x^{\varrho}\exp\left(-\frac{2x^2+y^2}{4x}\right), \ \Re\varrho \leq 1/2. \tag{7.16}$$

The condition $\Re\varrho \leq 1/2$ means that $2(1-\Re\varrho) \geq 1$ and consequently, from the above assumption we find that $\nu < 1$. Consider the index operator

$$(Ig)(x) = \int_0^\infty \tau \sinh(2\pi\tau)\Gamma\left(\frac{1}{2}-\varrho+i\tau\right)\Gamma\left(\frac{1}{2}-\varrho-i\tau\right)W_{\varrho,i\tau}(x)g(\tau)d\tau \tag{7.17}$$

with respect to an arbitrary function g. Using the Stirling formula (1.33) and asymptotic expansion (1.172) of the Whittaker function by its second index one can easily show that for each $x > 0$

$$\tau \sinh(2\pi\tau)\Gamma\left(\frac{1}{2}-\varrho+i\tau\right)\Gamma\left(\frac{1}{2}-\varrho-i\tau\right)W_{\varrho,i\tau}(x) = O\left(\tau^{1/2-\varrho}e^{\pi\tau/2}\right), \ \tau \to \infty. \tag{7.18}$$

Therefore under condition $g \in L_1\left(\mathbf{R}_+;\tau^{1/2-\varrho}e^{\pi\tau/2}\right)$ operator (7.17) exists in view of evident estimate

$$|(Ig)(x)| \leq C_x \int_0^\infty \tau^{1/2-\varrho}e^{\pi\tau/2}|g(\tau)|d\tau, \tag{7.19}$$

where a positive constant C_x depends from x. Further, the above assumptions enable to estimate composition (7.14). Indeed, invoking with inequality (1.100) after substitution representation (7.15) within (7.14) we obtain the following estimate

$$|W_{\varrho,i\tau}[f]| \leq \frac{2^\nu e^{-2\delta\tau}}{2\pi}\int_0^\infty K_0(y\cos\delta)y^{-\nu}dy\int_{\nu-i\infty}^{\nu+i\infty}\left|\frac{f^*(1-s)}{\Gamma(1-\varrho-s/2)}ds\right|. \tag{7.20}$$

Here as usually $\delta \in [0,\pi/2)$ and moreover, integrals by y and s are convergent owing to condition $\nu < 1$ and our assumption above. Consequently, one can achieve the condition $W_{\varrho,i\tau}[f] \in L_1\left(\mathbf{R}_+;\tau^{1/2-\varrho}e^{\pi\tau/2}\right)$ choosing parameter δ from the interval $(\pi/4,\pi/2)$.

Let us evaluate the following composition

$$\left(IW_{\varrho,i\tau}[f]\right)(x) = \int_0^\infty \tau \sinh(2\pi\tau)\Gamma\left(\frac{1}{2}-\varrho+i\tau\right)\Gamma\left(\frac{1}{2}-\varrho-i\tau\right)$$

$$\times W_{\varrho,i\tau}(x)W_{\varrho,i\tau}[f]d\tau. \tag{7.21}$$

The above estimates perform to change the order of integration in (7.21) after substitution the value of the index transform (7.11) given by formula (7.14). Making use the identity (7.16) we arrive to the representation

$$\left(IW_{\varrho,i\tau}[f]\right)(x) = \pi^2 2^{2(\varrho-1)}x^{\varrho}e^{-x/2}\int_0^\infty y^{1-2\varrho}e^{-y^2/4x}f_1(y)dy. \tag{7.22}$$

Since by our assumption the Mellin transform satisfies the condition

$$f^*(1-\nu-it) \in L_1\left(\mathbf{R}; [\Gamma(1-\varrho-(\nu+it)/2)]^{-1}\right),$$

then change again the order of integration substituting the value of $f_1(y)$ by formula (7.15). Thus, calculating the inner integral by y with the aid of Euler's integral (1.22) under condition $\Re(\varrho+\nu/2) < 1$ after simple interchange of variable $v = y^2/4x$ we obtain that

$$(IW_{\varrho,i\tau}[f])(x) = \frac{\pi x e^{-x/2}}{4i} \int_{\nu-i\infty}^{\nu+i\infty} f^*(1-s) x^{-s/2} ds. \tag{7.23}$$

However, as is obvious the Mellin transform $f^*(1-\nu-it) \in L_1(\mathbf{R})$ because of the above condition of the integrability with the weight $[\Gamma(1-\varrho-(\nu+it)/2)]^{-1}$. Thus the inversion of the Mellin transform gives us finally that

$$(IW_{\varrho,i\tau}[f])(x) = \frac{\pi^2 \sqrt{x} e^{-x/2}}{2} f\left(\frac{1}{\sqrt{x}}\right). \tag{7.24}$$

We summarize our results by the following theorem.

Theorem 7.3. *Let $f \in L_{1-\nu,p}(\mathbf{R}_+)$ with $1 < p \le 2$ and $\nu < 1$. Furthermore, let the Mellin transform $f^*(1-\nu-it)$ belong to the space $L_1\left(\mathbf{R}; [\Gamma(1-\varrho-(\nu+it)/2)]^{-1}\right)$. Then the index transform $g(\tau) = W_{\varrho,i\tau}[f]$, $\varrho \le 1/2, \tau \in \mathbf{R}_+$ with the Whittaker function as the kernel given by formula* (7.11) *exists and can be represented by composition* (7.14). *Moreover, for each $x > 0$ its dual formula of kind*

$$f\left(\frac{1}{\sqrt{x}}\right) = \frac{2e^{x/2}}{\pi^2\sqrt{x}}$$

$$\times \int_0^\infty \tau \sinh(2\pi\tau) \Gamma\left(\frac{1}{2}-\varrho+i\tau\right) \Gamma\left(\frac{1}{2}-\varrho-i\tau\right) W_{\varrho,i\tau}(x) g(\tau) d\tau \tag{7.25}$$

holds.

Remark 7.1. After replacement $v^{-2} = x$ in formula (7.11) and functional substitution $f(x)$ instead of $e^{-x/2}[2x^{3/2}]^{-1} f(x^{-1/2})$ in (7.25) we immediately arrive to the pair of reciprocal formulae of the index transform with a Whittaker function generated by expansion (1.234).

Our purpose now is to consider one direct generalization of the Olevskii transform (7.1) which contains special case of the Appel F_3-function as the kernel. We called this transform as F_3-*transform* (see the references at the beginning of this chapter). As is known for example in Erdélyi et al. [1] the Appel F_3-function of two variables is defined by the following double series

$$F_3(a, a_1, b, b_1; c; x, y) = \sum_{k,m=0}^{\infty} \frac{(a)_k (a_1)_m (b)_k (b_1)_m}{(c)_{k+m}} \frac{x^k y^m}{k! m!} \tag{7.26}$$

being convergent absolutely in the domain $|x|, |y| < 1$. However, as is shown in Marichev [1] the special case of the Appel function (7.26) $(x-1)^c F_3(a, a_1, b, b_1; c; 1-x, 1-x^{-1})$ with power multiplier for $x > 1$ can be expressed in terms of Meijer's G-function owing to Slater's theorem. Indeed, as a corollary of the expression (1.140) we obtain the following Mellin-Barnes integral

$$H(x-1)(x-1)^{c-1} F_3\left(a, a_1, b, b_1; c; 1-x, 1-\frac{1}{x}\right)$$

$$= \frac{\Gamma(c)}{2\pi i} \int_{\nu-i\infty}^{\nu+i\infty} \frac{\Gamma(1-a_1-b_1-s)\Gamma(1+a-c-s)\Gamma(1+b-c-s)}{\Gamma(1-a_1-s)\Gamma(1-b_1-s)\Gamma(1+a+b-c-s)} x^{-s} ds, \tag{7.27}$$

provided that the parameters satisfy the conditions $\Re c > 0$, $\nu < 1 - \Re(a_1 + b_1)$, $1 + \Re(a-c)$, $1 + \Re(b-c)$ and $H(x)$ is the Heaviside function. To introduce the F_3-transform we slightly change representation (7.27) in accordance with properties (1.61)-(1.62) of Meijer's G-function and correspondence (1.207) of the Mellin transform. More precisely, let $c = 1-\alpha, \Re\alpha < 1$ and $a = (1-i\tau)/2 - \alpha, b = (1+i\tau)/2 - \alpha$. Then one can write the following relation

$$H(x-1)(x^2-1)^{-\alpha} F_3\left(\frac{1-i\tau}{2} - \alpha, a_1, \frac{1+i\tau}{2} - \alpha, b_1; 1-\alpha; 1-x^2, 1-\frac{1}{x^2}\right)$$

$$= \frac{\Gamma(1-\alpha)}{4\pi i} \int_{\nu-i\infty}^{\nu+i\infty} \Gamma\left(\frac{1-i\tau-s}{2}\right) \Gamma\left(\frac{1+i\tau-s}{2}\right) \Gamma\left(1-a_1-b_1-\frac{s}{2}\right)$$

$$\times \left[\Gamma\left(1-a_1-\frac{s}{2}\right) \Gamma\left(1-b_1-\frac{s}{2}\right) \Gamma\left(1-\alpha-\frac{s}{2}\right)\right]^{-1} x^{-s} ds. \tag{7.28}$$

Consequently, for $\tau \in \mathbf{R}_+$ define F_3-transform by the following formula

$$(F_3 f)(\tau) = \frac{1}{\Gamma(1-\alpha)} \int_1^\infty (v^2-1)^{-\alpha}$$

$$\times F_3\left(\frac{1-i\tau}{2} - \alpha, a_1, \frac{1+i\tau}{2} - \alpha, b_1; 1-\alpha; 1-v^2, 1-\frac{1}{v^2}\right) f(v) dv, \tag{7.29}$$

where we assume that $\Re\alpha < 1$ and $f \in L_{1-\nu,p}(\mathbf{R}_+)$, $1 < p \le 2$, $\nu > \max(0, -2\Re\alpha)$. Since the integrand in (7.28) related to F_3-transform (7.29) equals $O(|t|^{\Re\alpha-1})$, $t = \Im s \in \mathbf{R}$ owing to the Stirling formula (1.33), then it belongs to $L_p(\mathbf{R})$ iff $\Re\alpha < 1/q, q = p/(p-1)$. Therefore by virtue of the Mellin-Parseval formula (1.214) we find that

$$(F_3 f)(\tau) = \frac{1}{4\pi i} \int_{\nu-i\infty}^{\nu+i\infty} \Gamma\left(\frac{1-i\tau-s}{2}\right) \Gamma\left(\frac{1+i\tau-s}{2}\right) \Gamma\left(1-a_1-b_1-\frac{s}{2}\right)$$

$$\times \left[\Gamma\left(1-a_1-\frac{s}{2}\right) \Gamma\left(1-b_1-\frac{s}{2}\right) \Gamma\left(1-\alpha-\frac{s}{2}\right)\right]^{-1} f^*(1-s) ds. \tag{7.30}$$

Denoting by $F(s)$ the gamma-ratio

$$F(s) = \frac{\Gamma(1-a_1-b_1-s/2)}{\Gamma(1-a_1-s/2)\Gamma(1-b_1-s/2)\Gamma(1-\alpha-s/2)} \tag{7.31}$$

and by $f_1(x), x > 0$ the function of type

$$f_1(x) = \frac{1}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty} 2^s F(s) f^*(1-s) x^{-s} ds. \tag{7.32}$$

As is easily seen $F(s) = O(e^{\pi|\Im s|/2}|s|^{\nu+\Re\alpha-1})$, $|\Im s| \to \infty$. Hence owing to representation (2.125) under condition

$$f^*(1-\nu-it) \in L_1(\mathbf{R}; e^{\pi|t|/2}|t|^{\nu+\Re\alpha-1})$$

from the Mellin-Parseval equality (1.214) we obtain that

$$(F_3 f)(\tau) = K_{i\tau}[f_1] = \int_0^\infty K_{i\tau}(y) f_1(y) dy. \tag{7.33}$$

Let us call now one auxiliary index integral given in Prudnikov et al. [2] by formula 2.16.49.1

$$\int_0^\infty \tau \sinh(\pi\tau)\Gamma\left(\frac{s+i\tau}{2}\right)\Gamma\left(\frac{s-i\tau}{2}\right) K_{i\tau}(y) d\tau = \pi^2 2^{1-s} y^s. \tag{7.34}$$

As is shown in Chapter 1 for each $y > 0$ the Macdonald function $K_{i\tau}(y)$ behaves asymptotically by formula (1.148) when $\tau \to \infty$. Hence invoking with Stirling's formula (1.33) we obtain that the integrand in (7.34) equals $O(\tau^{\nu-1/2}), \tau \to \infty$, where we mean as usually $\nu = \Re s$. This implies that integral (7.34) is conditionally convergent under assumption $0 < \nu < 1/2$ because of the presence the oscillation multipliers in asymptotic expansions. On the other hand, the left-hand side of (7.34) can be written almost everywhere on $\mathbf{R}_+$ as the following limit

$$\lim_{\varepsilon\to 0+}\int_0^\infty \tau \sinh((\pi-\varepsilon)\tau)\Gamma\left(\frac{s+i\tau}{2}\right)\Gamma\left(\frac{s-i\tau}{2}\right) K_{i\tau}(y) d\tau$$
$$= \lim_{\varepsilon\to 0+} I(y,\varepsilon). \tag{7.35}$$

Recalling again identity (2.125) substitute it in (7.35). Taking into account the value of the integral (2.17) after changing the order of integration we obtain

$$\lim_{\varepsilon\to 0+} I(y,\varepsilon) = 2^{1-s}\pi y \lim_{\varepsilon\to 0+} \sin\varepsilon \int_0^\infty v^s \frac{K_1((y^2+v^2-2vy\cos\varepsilon)^{1/2})}{(y^2+v^2-2vy\cos\varepsilon)^{1/2}} dv. \tag{7.36}$$

Of course, we performed the changing of the order of integration owing to Fubini's theorem. By the same treatment as in Theorem 2.2 we show that the limit in (7.36) exists under condition $0 < \nu < 1/2$ and equals to the right-hand side of (7.34). Furthermore, it gives us the uniform boundedness by $s = \nu + it, t \in \mathbf{R}$. Precisely, we change the variable $v = y(\cos\varepsilon + u\sin\varepsilon)$ and estimate the obtained integral as follows

$$|I(y,\varepsilon)| \le C\int_{-\infty}^\infty \frac{(1+|u|)^\nu}{u^2+1} du < \infty, \nu < 1, \tag{7.37}$$

where $C > 0$ is an absolute constant.

Now consider the following particular case of the Fox H-function (1.63)

$$H_{3,3}^{2,3}\left(x\left|\begin{matrix}(a_1,1/2),(b_1,1/2),(\alpha,1/2)\\(i\tau/2,1/2),(-i\tau/2,1/2),(a_1+b_1,1/2)\end{matrix}\right.\right)$$

$$=\frac{1}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty}\Gamma\left(\frac{s+i\tau}{2}\right)\Gamma\left(\frac{s-i\tau}{2}\right)\frac{x^{-s}}{F(s+1)}ds$$

$$=\frac{1}{2\pi i}\int_{\nu-i\infty}^{\nu+i\infty}\Gamma\left(\frac{s+i\tau}{2}\right)\Gamma\left(\frac{s-i\tau}{2}\right)$$

$$\times\frac{\Gamma((1-s)/2-a_1)\Gamma((1-s)/2-b_1)\Gamma((1-s)/2-\alpha)}{\Gamma((1-s)/2-a_1-b_1)}x^{-s}ds, \tag{7.39}$$

where the contour in (7.39) separates the series of the left and the right poles of gamma-functions. It can be achieved by the condition $0<\nu<\min(1-2a_1,1-2b_1,1-2\alpha)$. Hence combining (7.39) and (7.35) after integration through by s, changing the order by Fubini's theorem and the use of Lebesgue's theorem to carry out the limit we arrive to the following relation

$$\lim_{\varepsilon\to0+}\frac{1}{2\pi^2}\int_0^\infty\tau\sinh((\pi-\varepsilon)\tau)K_{i\tau}(y)$$

$$\times H_{3,3}^{2,3}\left(x\left|\begin{matrix}(a_1+1/2,1/2),(b_1+1/2,1/2),(\alpha+1/2,1/2)\\(i\tau/2,1/2),(-i\tau/2,1/2),(a_1+b_1+1/2,1/2)\end{matrix}\right.\right)d\tau$$

$$=H_{3,1}^{0,3}\left(\frac{2x}{y}\left|\begin{matrix}(a_1+1/2,1/2),(b_1+1/2,1/2),(\alpha+1/2,1/2)\\(a_1+b_1+1/2,1/2)\end{matrix}\right.\right). \tag{7.40}$$

Calculate now the composition of operators such that

$$I(x)=\lim_{\varepsilon\to0+}\frac{1}{2\pi^2}\int_0^\infty\tau\sinh((\pi-\varepsilon)\tau)$$

$$\times H_{3,3}^{2,3}\left(x\left|\begin{matrix}(a_1+1/2,1/2),(b_1+1/2,1/2),(\alpha+1/2,1/2)\\(i\tau/2,1/2),(-i\tau/2,1/2),(a_1+b_1+1/2,1/2)\end{matrix}\right.\right)(F_3f)(\tau)d\tau. \tag{7.41}$$

Substitute representation (7.33), change the order of integration and pass to the limit owing to the above assumptions. Making use (7.40) as a result it leads us to the equality

$$I(x)=\int_0^\infty f_1(y)$$

$$\times H_{3,1}^{0,3}\left(\frac{2x}{y}\left|\begin{matrix}(a_1+1/2,1/2),(b_1+1/2,1/2),(\alpha+1/2,1/2)\\(a_1+b_1+1/2,1/2)\end{matrix}\right.\right)dy. \tag{7.42}$$

Recall formula (7.32) after substitution within (7.42) and apply the Mellin-Parseval equality being slightly different from (1.214). We find that

$$I(x) = \frac{1}{\pi i}\int_{\nu-i\infty}^{\nu+i\infty} f^*(1-s)x^{-s}ds = \frac{2}{x}f\left(\frac{1}{x}\right). \tag{7.43}$$

Thus under the above assumptions we arrive to the following pair of reciprocal formulae of the F_3-transform, namely

$$(F_3 f)(\tau) = \frac{1}{\Gamma(1-\alpha)}\int_1^\infty (v^2-1)^{-\alpha}$$

$$\times F_3\left(\frac{1-i\tau}{2}-\alpha, a_1, \frac{1+i\tau}{2}-\alpha, b_1; 1-\alpha; 1-v^2, 1-\frac{1}{v^2}\right) f(v)dv,$$

$$f(x) = \lim_{\varepsilon\to 0+}\frac{1}{4\pi^2 x}\int_0^\infty \tau\sinh((\pi-\varepsilon)\tau)$$

$$\times H_{3,3}^{2,3}\left(\frac{1}{x}\middle| \begin{matrix}(a_1+1/2,1/2),(b_1+1/2,1/2),(\alpha+1/2,1/2)\\ (i\tau/2,1/2),(-i\tau/2,1/2),(a_1+b_1+1/2,1/2)\end{matrix}\right)(F_3 f)(\tau)d\tau, \tag{7.44}$$

where we need the values of $x \ge 1$. It is easy to show with the aid of Slater's theorem that the H-function in formula (7.44) can be reduced to the sum of two ${}_3F_2$-functions with symmetric parameters $\pm i\tau$.

7.2 General $\Re$-transforms

In this section following Yakubovich et al. [1,1994] we deal with certain generalization of the Lebedev-Skalskaya transforms being considered in the previous chapter. Namely, we introduce the general $\Re$-transforms that contain the kernels in terms of real (or imaginary) parts of the Meijer G-function by fixed complex index. In particular, we attract our attention to some examples of general expansions with the Bessel function kernels.

We need to give here some additional notions and facts concerning the theory of the Mellin convolution type integral transforms as well as several new relations between Meijer's G-functions and some of their particular cases. We shall use it for proving representation theorems for the introduced index transforms. First we give a definition of the G-transform being mentioned in Chapter 5. More detail information see, for example in Samko et al. [1].

Definition 7.1. The G-transform of a function $f(x)$ is defined by the integral

$$[Gf](x) \equiv \left(G_{p,q}^{m,n}\left| \begin{matrix}(\alpha_p)\\ \\ (\beta_q)\end{matrix}\right| f(t)\right)(x) = \frac{1}{2\pi i}\int_\sigma \Psi(s)f^*(s)x^{-s}ds, \quad x>0, \tag{7.45}$$

where the kernel $\Psi(s)$ defined by the gamma-ratio (1.60), $f^*(s)$ is the Mellin transform (1.204) of the function f, σ is the contour $\{s; \Re s = \frac{1}{2}\}$ on the complex s-plane, and vectors (α_p) and (β_q) satisfy the condition

$$\begin{cases} \Re\beta_j > -\frac{1}{2}, \quad j=1,\cdots,m; & \Re\alpha_j < \frac{1}{2}, \quad j=1,\cdots,n; \\ \Re\alpha_j > -\frac{1}{2}, \quad j=n+1,\cdots,p; & \Re\beta_j < \frac{1}{2}, \quad j=m+1,\cdots,q. \end{cases} \tag{7.46}$$

Well regulated pair (c^*, γ^*) with

$$c^* = m + n - \frac{p+q}{2}, \quad \gamma^* = \Re\left(\sum_{j=1}^{p} \alpha_j - \sum_{j=1}^{q} \beta_j\right) \tag{7.47}$$

is called the characteristic of the G-transform (7.45).

Remark 7.2. As one can conclude from Definition 7.1 the operators used above in compositions with the Kontorovich-Lebedev transform to invert more general index transforms in L_p-spaces are similar to the G-transform (7.45). Nevertheless, we distinguish it here as a separate case in view of the contour of integration σ and special space of functions being defined below which relates to the G-transform (7.45).

Definition 7.2. Let c, γ be real numbers with

$$2\operatorname{sign} c + \operatorname{sign} \gamma \geq 0. \tag{7.48}$$

The space of functions $f(x)$ of the form

$$f(x) = \frac{1}{2\pi i} \int_\sigma f^*(s) x^{-s} ds, \tag{7.49}$$

with

$$f^*(s)|s|^\gamma e^{\pi c|\Im s|} \in L(\sigma) \tag{7.50}$$

is denoted by $\mathcal{M}_{c,\gamma}^{-1}(L)$, where $\sigma = \{s; \Re s = \frac{1}{2}\}$.

The space $\mathcal{M}_{c,\gamma}^{-1}(L)$ with the norm

$$\| f \|_{\mathcal{M}_{c,\gamma}^{-1}(L)} = \int_\sigma |s|^\gamma e^{\pi c|\Im s|} |f^*(s) ds| \tag{7.51}$$

is a Banach space.

These notions of the G-transform and special suitable space of functions are described precisely for instance in Samko et al. [1], Vu Kim Tuan et al. [1], Nguyen Thanh Hai and Yakubovich [1]. Here we give certain properties of the G-transform (7.45) in the space $\mathcal{M}_{c,\gamma}^{-1}(L)$ which shall be useful for our considerations below.

Theorem 7.4. *The G-transform* (7.45) *with the characteristic* (c^*, γ^*) *exists in the space* $\mathcal{M}_{c,\gamma}^{-1}(L)$ *if and only if the next inequality*

$$2\operatorname{sign}(c + c^*) + \operatorname{sign}(\gamma + \gamma^*) \geq 0 \tag{7.52}$$

holds. Moreover, this G-transform is an isomorphism of the space $\mathcal{M}^{-1}_{c,\gamma}(L)$ *onto the space* $\mathcal{M}^{-1}_{c+c^*,\gamma+\gamma^*}(L)$.

Theorem 7.5. *Let* $f(x)$ *be from the space* $\mathcal{M}^{-1}_{c,\gamma}(L)$ *and inequality* (7.52) *be valid as well as*

$$2\operatorname{sign} c^* + \operatorname{sign}(\gamma^* + 1) < 0. \tag{7.53}$$

Then if the G-transform (7.45) $[Gf](x)$ *be from the space* $L_1(\mathbf{R}_+; x^{-1/2})$, *its inversion formula is given by*

$$f(x) = \int_0^\infty G^{p-n,q-m}_{q,p}\left(\frac{x}{y}\,\middle|\,\begin{matrix}(\beta_q^{m+1}),(\beta_m)\\ \\ (\alpha_p^{n+1}),(\alpha_n)\end{matrix}\right)[Gf](y)\frac{dy}{y}, \tag{7.54}$$

where the integral is absolutely convergent.

Proof. According to the Definition 7.1 it is clear that $[Gf](x)$ is the inverse Mellin transform (7.49) of the product $\Psi(s)f^*(s)$ along the contour σ and due to the Stirling formula (1.32) for gamma-function we have that $\Psi(s) \sim |\Im s|^{-\gamma^*}e^{-|\Im s|\pi c^*}$, as $|\Im s| \to \infty$. Hence by condition (7.52) we conclude that integral (7.45) is absolutely convergent and we have

$$|[Gf](x)| \le \frac{x^{-1/2}}{2\pi}\int_\sigma |\Psi(s)f^*(s)ds| = Ax^{-1/2}. \tag{7.55}$$

Further, inequality (7.53) allows us to observe that integral (1.59) for the G-function within the right-hand side of (7.54) is absolutely convergent too. In fact, the kernel of this G-function is $\Psi^{-1}(s)$ and it behaves as $|\Im s|^{\gamma^*}e^{|\Im s|\pi c^*}$ when $|\Im s| \to \infty$. Since $[Gf](x)$ is from the space $L_1(\mathbf{R}_+; x^{-1/2})$, then we have the estimate

$$\int_0^\infty \left|G^{p-n,q-m}_{q,p}\left(\frac{x}{y}\,\middle|\,\begin{matrix}(\beta_q^{m+1}),(\beta_m)\\ \\ (\alpha_p^{n+1}),(\alpha_n)\end{matrix}\right)[Gf](y)\right|\frac{dy}{y}$$

$$\le x^{-1/2}\int_0^\infty |[Gf](y)|y^{-1/2}\frac{1}{2\pi}\int_\sigma\left|\frac{dsdy}{\Psi(s)}\right| < +\infty. \tag{7.56}$$

Therefore we apply the Fubini theorem to change the order of integration in the right-hand side of equality (7.54) after substituting representation (1.59) for the respective G-function. For this we need to evaluate the inner integral

$$\int_0^\infty [Gf](y)y^{s-1}dy, \tag{7.57}$$

which gives the value $\Psi(s)f^*(s)$ according to the inverse theorem for the Mellin transform (see, for example Titchmarsh [1]), when both of the function and its Mellin image are absolutely integrable. So we have finally from the right-hand side of (7.54) the equalities

$$\int_0^\infty G^{p-n,q-m}_{q,p}\left(\frac{x}{y}\,\middle|\,\begin{matrix}(\beta_q^{m+1}),(\beta_m)\\ \\ (\alpha_p^{n+1}),(\alpha_n)\end{matrix}\right)[Gf](y)\frac{dy}{y}$$

$$= \frac{1}{2\pi i}\int_\sigma \frac{1}{\Psi(s)} x^{-s}\int_0^\infty [Gf](y)y^{s-1}dyds$$
$$= \frac{1}{2\pi i}\int_\sigma \frac{1}{\Psi(s)}\Psi(s)f^*(s)x^{-s}ds = f(x). \tag{7.58}$$

This gives the desired result for the function f of $\mathcal{M}_{c,\gamma}^{-1}(L)$. Thus the proof of Theorem 7.5 is completed. •

Now let us consider some simple identities between G-functions (1.59) and their particular cases to complete the list of formulae (1.107)-(1.140) and use in our further discussions for constructions the general $\Re$-transforms. At first it is not difficult to check with the reduction formula (1.23) for Euler's gamma-function the following identities

$$\Gamma(a-b-1/2)\Gamma(a+b+1/2)+\Gamma(a+b-1/2)\Gamma(a-b+1/2)$$
$$= (2a-1)\Gamma(a-b-1/2)\Gamma(a+b-1/2), \tag{7.59}$$
$$\Gamma(a-b-1/2)\Gamma(a+b+1/2)-\Gamma(a+b-1/2)\Gamma(a-b+1/2)$$
$$= 2b\Gamma(a-b-1/2)\Gamma(a+b-1/2), \tag{7.60}$$

where a, b are certain complex numbers. Hence making use identity (7.59), we have the equality

$$\Re_{i\tau} G^{m,n+2}_{p+2,q}\left(x \left| \begin{matrix} 1/4+i\tau, -3/4-i\tau, (\alpha_p) \\ \\ (\beta_q) \end{matrix}\right.\right)$$
$$= G^{m,n+3}_{p+3,q+1}\left(x \left| \begin{matrix} 1/4+i\tau, 1/4-i\tau, -3/4, (\alpha_p) \\ \\ (\beta_q), 1/4 \end{matrix}\right.\right). \tag{7.61}$$

Indeed, taking account of (1.59) and putting in (7.59) $a = 5/4 - s, b = i\tau$, we have

$$\Re_{i\tau} G^{m,n+2}_{p+2,q}\left(x \left| \begin{matrix} 1/4+i\tau, -3/4-i\tau, (\alpha_p) \\ \\ (\beta_q) \end{matrix}\right.\right)$$
$$= \frac{1}{2\pi i}\Re_{i\tau}\int_L \Psi(s)\Gamma\left(\frac{3}{4}-i\tau-s\right)\Gamma\left(\frac{7}{4}+i\tau-s\right)x^{-s}ds$$
$$= \frac{1}{4\pi i}\int_L \Psi(s)\left[\Gamma\left(\frac{3}{4}-i\tau-s\right)\Gamma\left(\frac{7}{4}+i\tau-s\right)\right.$$
$$\left.+\Gamma\left(\frac{3}{4}+i\tau-s\right)\Gamma\left(\frac{7}{4}-i\tau-s\right)\right]x^{-s}ds$$
$$= \frac{1}{2\pi i}\int_L \Psi(s)\Gamma\left(\frac{3}{4}-i\tau-s\right)\Gamma\left(\frac{3}{4}+i\tau-s\right)\frac{\Gamma(7/4-s)}{\Gamma(3/4-s)}x^{-s}ds$$
$$= G^{m,n+3}_{p+3,q+1}\left(x \left| \begin{matrix} 1/4+i\tau, 1/4-i\tau, -3/4, (\alpha_p) \\ \\ (\beta_q), 1/4 \end{matrix}\right.\right). \tag{7.62}$$

Similarly, using identity (7.60) we find

$$\Im_{i\tau} G^{m,n+2}_{p+2,q}\left(x \left| \begin{matrix} -1/4 - i\tau, 3/4 + i\tau, (\alpha_p) \\ \\ (\beta_q) \end{matrix} \right.\right)$$

$$= \tau G^{m,n+2}_{p+2,q}\left(x \left| \begin{matrix} 3/4 - i\tau, 3/4 + i\tau, (\alpha_p) \\ \\ (\beta_q) \end{matrix} \right.\right). \tag{7.63}$$

More precisely, letting $a = 3/4 - s, b = i\tau$ in (7.60) we have

$$\Im_{i\tau} G^{m,n+2}_{p+2,q}\left(x \left| \begin{matrix} -1/4 - i\tau, 3/4 + i\tau, (\alpha_p) \\ \\ (\beta_q) \end{matrix} \right.\right)$$

$$= \frac{1}{4\pi} \int_L \Psi(s) \left[\Gamma\left(\frac{5}{4} - i\tau - s\right)\Gamma\left(\frac{1}{4} + i\tau - s\right)\right.$$

$$\left. -\Gamma\left(\frac{5}{4} + i\tau - s\right)\Gamma\left(\frac{1}{4} - i\tau - s\right)\right] x^{-s} ds$$

$$= \frac{\tau}{2\pi i} \int_L \Psi(s)\Gamma\left(\frac{1}{4} - i\tau - s\right)\Gamma\left(\frac{1}{4} + i\tau - s\right) x^{-s} ds$$

$$= \tau G^{m,n+2}_{p+2,q}\left(x \left| \begin{matrix} 3/4 + i\tau, 3/4 - i\tau, (\alpha_p) \\ \\ (\beta_q) \end{matrix} \right.\right). \tag{7.64}$$

By this way, combining some particular cases of G-functions we can complete the list of formulae in Chapter 1. For instance, taking relation (1.117) with the square of the Macdonald function owing to the reflection formula (1.61) for the respective G-function and the obtained identity (7.59) we corresponds the following equality

$$\Re K^2_{1/2+i\tau}\left(\frac{1}{\sqrt{x}}\right) = \frac{\sqrt{\pi}}{2} G^{0,3}_{3,1}\left(x \left| \begin{matrix} 3/2 - i\tau, 3/2 + i\tau, 1 \\ \\ 3/2 \end{matrix} \right.\right). \tag{7.65}$$

We reduce below the list of such useful new relations of $\Re$-functions for $\Re$-index transforms

$$\Re\left[K_{i\tau}\left(\frac{1}{\sqrt{x}}\right) K_{1+i\tau}\left(\frac{1}{\sqrt{x}}\right)\right] = \frac{\sqrt{\pi}}{2} G^{0,3}_{3,1}\left(x \left| \begin{matrix} 3/2 - i\tau, 3/2 + i\tau, 1/2 \\ \\ 1 \end{matrix} \right.\right), \tag{7.66}$$

$$e^{-1/2x} \Re K_{1/2+i\tau}\left(\frac{1}{2x}\right) = \sqrt{\pi} G^{0,2}_{2,1}\left(x \left| \begin{matrix} 3/2 - i\tau, 3/2 + i\tau \\ \\ 3/2 \end{matrix} \right.\right), \tag{7.67}$$

$$e^{1/2x} \Re K_{1/2+i\tau}\left(\frac{1}{2x}\right) = \frac{\tau \sinh(\pi\tau)}{\sqrt{\pi}} G^{1,2}_{2,1}\left(x \left| \begin{matrix} 3/2 - i\tau, 3/2 + i\tau \\ \\ 1/2 \end{matrix} \right.\right), \tau > 0, \tag{7.68}$$

$$\Re\left[K_{\alpha+i\tau}\left(\frac{1}{\sqrt{x}}\right)K_{\beta+i\tau}\left(\frac{1}{\sqrt{x}}\right)\right]$$

$$= \frac{\sqrt{\pi}}{2}G^{0,4}_{4,2}\left(x\,\middle|\,\begin{array}{l}3/2-i\tau, 3/2+i\tau, 1+(\alpha-\beta)/2, 1+(\beta-\alpha)/2\\ \\ 1, 3/2\end{array}\right), \tag{7.69}$$

where $\alpha+\beta=1$,

$$\Re_{i\tau}\left[\Gamma(1+i\tau-\mu)\Gamma(-\mu-i\tau)(1+x)^{\mu/2}P^{\mu}_{i\tau}(1+2x)\right]$$

$$= G^{1,2}_{2,2}\left(x\,\middle|\,\begin{array}{l}1-i\tau+\mu/2, 1+i\tau+\mu/2,\\ \\ -\mu/2, 1+\mu/2\end{array}\right), \tag{7.70}$$

where $P^{\mu}_{i\tau}(1+2x)$ is the Legendre function (1.55) and

$$\Re_{i\tau}\left[(1+x)^{-\mu/2}P^{\mu}_{i\tau}(1+2x)\right]$$

$$= \frac{\tau\sinh(\pi\tau)}{\pi}G^{1,2}_{2,2}\left(x\,\middle|\,\begin{array}{l}1-i\tau-\mu/2, 1+i\tau-\mu/2,\\ \\ -\mu/2, \mu/2\end{array}\right), \tau>0. \tag{7.71}$$

To establish this list we need some additional discussions for which we take up for instance, formula (7.68). To write its right-hand side we need to adopt as the contour the right infinite loop for the respective G-function (see the description under formula (1.60)), because there exists no straight line to separate three series of poles. Indeed, we have

$$G^{1,2}_{2,1}\left(x\,\middle|\,\begin{array}{l}3/2-i\tau, 3/2+i\tau\\ \\ 1/2\end{array}\right)$$

$$= \frac{1}{2\pi i}\int_{L_\infty}\Gamma\left(\frac{1}{2}+s\right)\Gamma\left(-\frac{1}{2}-i\tau-s\right)\Gamma\left(-\frac{1}{2}+i\tau-s\right)x^{-s}ds, \tag{7.72}$$

where the right loop L_∞ separates the right series of the poles $s=-1/2-i\tau+m$, $m=0,1,\ldots$, $s=-1/2+i\tau+n$, $n=0,1,\ldots$, from the left ones $s=-1/2-k$, $k=0,1,\ldots$. Applying identity (7.59) and dividing (7.72) into two contour integrals, we obtain

$$G^{1,2}_{2,1}\left(x\,\middle|\,\begin{array}{l}3/2-i\tau, 3/2+i\tau\\ \\ 1/2\end{array}\right)$$

$$= -\frac{1}{4\pi\tau}\int_{L_\infty}\Gamma\left(\frac{1}{2}+s\right)\Gamma\left(-\frac{1}{2}-i\tau-s\right)\Gamma\left(\frac{1}{2}+i\tau-s\right)x^{-s}ds$$

$$+\frac{1}{4\pi\tau}\int_{L_\infty}\Gamma\left(\frac{1}{2}+s\right)\Gamma\left(-\frac{1}{2}+i\tau-s\right)\Gamma\left(\frac{1}{2}-i\tau-s\right)x^{-s}ds. \tag{7.73}$$

The last integrals of the Mellin-Barnes type in (7.73) can be evaluated by spreading the reflected equality (1.115), precisely

$$e^{(2x)^{-1}}K_{\mu}\left(\frac{1}{2x}\right)=\frac{\cos(\pi\mu)}{2\pi i\sqrt{\pi}}\int_{\nu-i\infty}^{\nu+i\infty}\Gamma\left(s+\frac{1}{2}\right)\Gamma\left(\mu-s\right)\Gamma\left(-\mu-s\right)x^{-s}ds, \qquad (7.74)$$

where $-1/2<\nu<-|\Re\mu|$ on the loop's type contour. Changing the contour in (7.74) to the respective right loop by the analytic properties of the integrand, putting $\mu=1/2+i\tau$ one can easily arrive to (7.68). Similarly we can establish the other formulae exhibited above.

We start now to consider one general index transform that involves G-function (1.59) and generates the expansion of an arbitrary function f similar to (1.237). Namely, we introduce the general $\Re$-transform of kind

$$(\Re G f)(\tau)=\int_0^{\infty}\Re_{i\tau}G^{m,n+2}_{p+2,q}\left(y\,\middle|\,\begin{array}{c}1/4+i\tau,-3/4-i\tau,(\alpha_p)\\ \\ (\beta_q)\end{array}\right)f(y)dy, \qquad (7.75)$$

where $\tau\geq 0$. The key purpose is to establish sufficient conditions for the validity of the following expansion

$$f(x)=\lim_{\varepsilon\to 0+}\frac{1}{\pi^2}\int_0^{\infty}\tau\sinh(2\pi\tau)$$

$$\times\Re_{i\tau}G^{q-m+1,p-n+3}_{p+4,q+2}\left(x\,\middle|\,\begin{array}{c}3/4-\varepsilon+i\tau,-1/4-\varepsilon-i\tau,3/4-\varepsilon,-(\alpha_p^{n+1}),-(\alpha_n),3/4\\ \\ -1/4,-(\beta_q^{m+1}),-(\beta_m),-1/4-\varepsilon\end{array}\right)$$

$$\times\int_0^{\infty}\Re_{i\tau}G^{m,n+2}_{p+2,q}\left(y\,\middle|\,\begin{array}{c}1/4+i\tau,-3/4-i\tau,(\alpha_p)\\ \\ (\beta_q)\end{array}\right)f(y)dyd\tau,\ \ x>0, \qquad (7.76)$$

which leads to the inversion formula for the general index $\Re$-transform (7.75) as

$$f(x)=\lim_{\varepsilon\to 0+}\frac{1}{\pi^2}\int_0^{\infty}\tau\sinh(2\pi\tau)$$

$$\times\Re_{i\tau}G^{q-m+1,p-n+3}_{p+4,q+2}\left(x\,\middle|\,\begin{array}{c}3/4-\varepsilon+i\tau,-1/4-\varepsilon-i\tau,3/4-\varepsilon,-(\alpha_p^{n+1}),-(\alpha_n),3/4\\ \\ -1/4,-(\beta_q^{m+1}),-(\beta_m),-1/4-\varepsilon\end{array}\right)$$

$$\times(\Re G f)(\tau)d\tau. \qquad (7.77)$$

We are ready to prove the following theorem.

Theorem 7.6. *Let $f(x)$ be $\mathcal{M}^{-1}_{c,\gamma}(L)\cap L_1(\mathbf{R}_+;x^{-1/2})$ and the G-transform*

$$\left(G^{m,n+1}_{p+2,q+1}\,\middle|\,\begin{array}{c}-3/4,(\alpha_p),-1/4\\ \\ (\beta_q),1/4\end{array}\,\middle|\,\frac{1}{t}f\left(\frac{1}{t}\right)\right)(x)$$

$$= \frac{1}{2\pi i}\int_{\sigma}\frac{\Gamma(7/4-s)}{\Gamma(3/4-s)\Gamma(-1/4+s)}\Psi(s)f^{*}(1-s)x^{s}ds, x>0, \tag{7.78}$$

be from $L_1(\mathbf{R}_+; e^{(2x)^{-1}}x^{-3/2})$. *Moreover assume that the system of inequalities*

$$2\mathrm{sign}\left(c+c^{*}-\frac{1}{2}\right)+\mathrm{sign}\left(\gamma+\gamma^{*}-\frac{5}{4}\right)>0, \tag{7.79}$$

$$2\mathrm{sign}\,(c^{*}+1)+\mathrm{sign}\left(\gamma^{*}-\frac{3}{2}\right)>0, \tag{7.80}$$

$$2\mathrm{sign}\, c^{*}+\mathrm{sign}\left(\gamma^{*}+\left(\frac{1}{2}-\xi\right)(p-q)\right)<0, \tag{7.81}$$

holds valid, where $1/4<\xi<1/4+\varepsilon, \varepsilon>0$ *and parameters* (c^{*},γ^{*}), (c,γ) *are defined by formulae* (7.47) – (7.48), *respectively. Then under the conditions on parameters of the kernel*

$$\begin{cases} \Re\beta_j > -\frac{1}{2}, \ \ j=1,\cdots,m; & \Re\alpha_j<\frac{1}{2}, \ \ j=1,\cdots,n; \\ \Re\alpha_j>-\frac{1}{2}, \ \ j=n+1,\cdots,p; & \Re\beta_j<\frac{1}{4}, \ \ j=m+1,\cdots,q. \end{cases}, \tag{7.82}$$

expansion (7.76) *takes place for each point* $x>0$.

Proof. By making use of identity (7.59) it is not difficult to establish the relation

$$\Re_{i\tau}G^{q-m+1,p-n+3}_{p+4,q+2}\left(x\,\middle|\,\begin{array}{l}3/4-\varepsilon+i\tau,-1/4-\varepsilon-i\tau,3/4-\varepsilon,-(\alpha_p^{n+1}),-(\alpha_n),3/4\\ -1/4,-(\beta_q^{m+1}),-(\beta_m),-1/4-\varepsilon\end{array}\right)$$

$$=G^{q-m+1,p-n+2}_{p+3,q+1}\left(x\,\middle|\,\begin{array}{l}3/4-\varepsilon+i\tau,3/4-\varepsilon-i\tau,-(\alpha_p^{n+1}),-(\alpha_n),3/4\\ -1/4,-(b_q^{m+1}),-(b_m)\end{array}\right), \tag{7.83}$$

where the contour is chosen as the vertical line $L_\xi=(\xi-i\infty,\xi+i\infty), 1/4<\xi<1/4+\varepsilon,\varepsilon>0$ in view of (7.82). Denoting the iterated integral in the right-hand side of equality (7.76) as $I(x,\varepsilon)$ and using identities (7.61), (7.83), we write

$$I(x,\varepsilon)=\frac{1}{\pi^2}\int_0^\infty \tau\sinh(2\pi\tau)$$

$$\times G^{q-m+1,p-n+2}_{p+3,q+1}\left(x\,\middle|\,\begin{array}{l}3/4-\varepsilon+i\tau,3/4-\varepsilon-i\tau,-(\alpha_p^{n+1}),-(\alpha_n),3/4\\ -1/4,-(\beta_q^{m+1}),-(\beta_m)\end{array}\right)$$

$$\times\int_0^\infty G^{m,n+3}_{p+3,q+1}\left(y\,\middle|\,\begin{array}{l}1/4+i\tau,1/4-i\tau,-3/4,(\alpha_p)\\ (\beta_q),1/4\end{array}\right)f(y)dy. \tag{7.84}$$

The inner integral in (7.84) is already denoted as $(\Re Gf)(\tau)$ and we now need to prove the following integral representation holds valid

$$(\Re Gf)(\tau)=\frac{1}{2\pi i}\int_\sigma\frac{\Gamma(7/4-s)}{\Gamma(3/4-s)}\Psi(s)$$

$$\times \Gamma\left(\frac{3}{4} - i\tau - s\right)\Gamma\left(\frac{3}{4} + i\tau - s\right) f^*(1-s)ds. \tag{7.85}$$

Indeed, the integrand in (7.85) contains the integrand of the G-function of the introduced $\Re$-transform (7.75). We have the asymptotic $|\Im s|^{1/2-\gamma^*} e^{-|\Im s|\pi(c^*+1)}$ when $\Re s = 1/2$ and $|\Im s| \to \infty$ from Stirling's formula (1.33). Since the corresponding integral (1.59) for this G-function under conditions (7.82) contains singularity only at infinity, inequality (7.80) provides its absolute convergence and the condition $f \in L(\mathbf{R}_+; x^{-1/2})$ allows us to apply the Fubini theorem to change the order of integration in the iterated integral (7.75) after substitution instead of the G-function of its respective representation (1.59). This leads to equality (7.85).

On the other hand we use the value of the integral being easily deduced from relation (1.115) by changing the variable and parameters

$$\frac{\sqrt{\pi}}{\cosh(\pi\tau)} \int_0^\infty K_{i\tau}\left(\frac{1}{2y}\right) e^{(2y)^{-1}} y^{s-7/4} dy$$

$$= \Gamma\left(\frac{3}{4} - i\tau - s\right)\Gamma\left(\frac{3}{4} + i\tau - s\right)\Gamma\left(s - \frac{1}{4}\right) \tag{7.86}$$

in order to obtain the integral representation of $\Re$-transform (7.75) through the Kontorovich-Lebedev transform (2.1) and G-transform (7.45) as

$$(\Re G f)(\tau)$$

$$= \frac{\sqrt{\pi}}{\cosh(\pi\tau)} \int_0^\infty K_{i\tau}\left(\frac{1}{2y}\right) e^{(2y)^{-1}} y^{-7/4}$$

$$\times \left(G^{m,n+1}_{p+2,q+1} \left| \begin{matrix} -3/4, (\alpha_p), -1/4 \\ \\ (\beta_q), 1/4 \end{matrix} \right| \frac{1}{t} f\left(\frac{1}{t}\right) \right)(y) dy. \tag{7.87}$$

A similar treatment for the integrand in (7.78) implies the estimate

$$\left| \left(G^{m,n+1}_{p+2,q+1} \left| \begin{matrix} -3/4, (\alpha_p), -1/4 \\ \\ (\beta_q), 1/4 \end{matrix} \right| \frac{1}{t} f\left(\frac{1}{t}\right) \right)(x) \right|$$

$$\le \frac{x^{1/2}}{2\pi} \int_\sigma \left| \frac{\Gamma(7/4-s)}{\Gamma(3/4-s)\Gamma(-1/4+s)} \Psi(s) \right|$$

$$\times |s|^{-\gamma} e^{-\pi c|\Im s|} |s|^{\gamma} e^{\pi c|\Im s|} \, |f^*(1-s)ds|$$

$$\le A x^{1/2} \int_\sigma |s|^{-(\gamma+\gamma^*-5/4)} e^{-\pi(c+c^*-1/2)|\Im s|} |s|^{\gamma} e^{\pi c|\Im s|} |f^*(1-s)ds|$$

$$\le A_1 x^{1/2} ||f||_{\mathcal{M}^{-1}_{c,\gamma}(L)} < \infty, \tag{7.88}$$

by virtue of Definition 7.2 and relation (7.79), where A, A_1 are positive constants. Thus, substituting the value of integral (7.86) into representation (7.85) and applying the Fubini theorem and formula (7.78), we obtain (7.87). Here inequality (1.147) and the convergent integral

$$\int_0^\infty K_0\left(\frac{1}{2y}\right) e^{(2y)^{-1}} y^{-5/4} dy < \infty, \tag{7.89}$$

are applied. Consider now the G-function in the outside integral (7.84). Using formula (2.125) with simple interchanges arrive to the value of the integral

$$2\int_0^\infty K_{2i\tau}\left(\frac{2}{\sqrt{y}}\right) y^{s-5/4-\varepsilon} dy = \Gamma\left(\frac{1}{4}+\varepsilon-i\tau-s\right)\Gamma\left(\frac{1}{4}+\varepsilon+i\tau-s\right), \tag{7.90}$$

which, in turn, leads to the relation

$$G^{q-m+1,p-n+2}_{p+3,q+1}\left(x \left| \begin{matrix} 3/4-\varepsilon+i\tau, 3/4-\varepsilon-i\tau, -(\alpha_p^{n+1}), -(\alpha_n), 3/4 \\ -1/4, -(\beta_q^{m+1}), -(\beta_m) \end{matrix} \right.\right)$$

$$= \frac{1}{2\pi i}\int_{L_\xi} \frac{\Gamma(-1/4+s)}{\Gamma(3/4+s)\Psi(1-s)}\Gamma\left(\frac{1}{4}-i\tau+\varepsilon-s\right)\Gamma\left(\frac{1}{4}+i\tau+\varepsilon-s\right) x^{-s} ds$$

$$= 2\int_0^\infty K_{2i\tau}\left(\frac{2}{\sqrt{y}}\right) y^{-5/4-\varepsilon} G^{q-m+1,p-n}_{p+1,q+1}\left(\frac{x}{y} \left| \begin{matrix} -(\alpha_p^{n+1}), -(\alpha_n), 3/4 \\ -1/4, -(\beta_q^{m+1}), -(\beta_m) \end{matrix} \right.\right) dy, \tag{7.91}$$

where the contour L_ξ was announced with (7.83). Indeed, the last equality is obtained by changing the order of integration, because the integral on the contour L_ξ is absolutely convergent under condition (7.81) in view of Stirling's formula (1.33) and the convergent integral

$$\int_0^\infty K_0\left(\frac{2}{\sqrt{y}}\right) y^{\xi-5/4-\varepsilon} dy < \infty, \tag{7.92}$$

when $1/4 < \xi < \xi + \varepsilon$.

Further, substituting the obtained representations (7.87) and (7.91) into (7.84), we find that

$$I(x,\varepsilon) = \frac{2}{\pi^{3/2}}\int_0^\infty \tau\sinh(\pi\tau)\int_0^\infty K_{2i\tau}\left(\frac{2}{\sqrt{y}}\right) y^{-5/4-\varepsilon}$$

$$\times G^{q-m+1,p-n}_{p+1,q+1}\left(\frac{x}{y} \left| \begin{matrix} -(\alpha_p^{n+1}), -(\alpha_n), 3/4 \\ -1/4, -(\beta_q^{m+1}), -(\beta_m) \end{matrix} \right.\right) dy$$

$$\times \int_0^\infty K_{i\tau}\left(\frac{1}{2u}\right) e^{(2u)^{-1}} u^{-7/4}\left(G^{m,n+1}_{p+2,q+1}\left| \begin{matrix} -3/4, (\alpha_p), -1/4 \\ (\beta_q), 1/4 \end{matrix} \right| \frac{1}{t} f\left(\frac{1}{t}\right)\right)(u) du d\tau. \tag{7.93}$$

To change the order of integration (we omit details) in this iterated integral we need to use inequality (1.100) as well as the following estimates

$$|K_{i\tau}(x)| \le A\frac{x^{-1/4}}{\sqrt{\sinh(\pi\tau)}}, \quad x > 0, \tag{7.94}$$

where A is a positive constant (see Lebedev [1]),

$$G^{q-m+1,p-n}_{p+1,q+1}\left(\frac{x}{u}\left|\begin{array}{l} -(\alpha_p^{n+1}), -(\alpha_n), 3/4 \\ \\ -1/4, -(\beta_q^{m+1}), -(\beta_m)\end{array}\right.\right) = O\left(|u|^{1-\alpha}\right), u \to 0, \tag{7.95}$$

$$G^{q-m+1,p-n}_{p+1,q+1}\left(\frac{x}{u}\left|\begin{array}{l} -(\alpha_p^{n+1}), -(\alpha_n), 3/4 \\ \\ -1/4, -(\beta_q^{m+1}), -(\beta_m)\end{array}\right.\right) = O\left(|u|^{-\beta}\right), u \to \infty, \tag{7.96}$$

where $\alpha = -\min_{n+1<k\le p} \Re\alpha_k$ and $\beta = \min(-\max_{m+1\le k\le q} \Re\beta_k, -1/4)$. By virtue of the inequalities (7.83) and the condition for the G-transform (7.78) being from the space $L_1(\mathbf{R}_+; e^{(2x)^{-1}}x^{-3/2})$ we are led to the expression

$$I(x,\varepsilon) = \frac{2}{\pi^{3/2}}\int_0^\infty e^{(2y)^{-1}} y^{-7/4}\left(G^{m,n+1}_{p+2,q+1}\left|\begin{array}{l} -3/4, (\alpha_p), -1/4 \\ \\ (\beta_q), 1/4\end{array}\right|\frac{1}{t}f\left(\frac{1}{t}\right)\right)(y)$$

$$\times\int_0^\infty G^{q-m+1,p-n}_{p+1,q+1}\left(\frac{x}{u}\left|\begin{array}{l} -(\alpha_p^{n+1}), -(\alpha_n), 3/4 \\ \\ -1/4, -(\beta_q^{m+1}), -(\beta_m)\end{array}\right.\right) u^{-5/4-\varepsilon}$$

$$\times\int_0^\infty \tau\sinh(\pi\tau)K_{i\tau}\left(\frac{1}{2y}\right)K_{2i\tau}\left(\frac{2}{\sqrt{u}}\right)d\tau du dy. \tag{7.97}$$

But the inner integral by τ in (7.97) is evaluated by formula 2.16.51.9 in Prudnikov et al. [2]

$$\int_0^\infty \tau\sinh(\pi\tau)K_{i\tau}\left(\frac{1}{2y}\right)K_{2i\tau}\left(\frac{2}{\sqrt{u}}\right)d\tau = \frac{\pi^{3/2}}{2}\sqrt{\frac{y}{u}}e^{-(2y)^{-1}-\frac{y}{u}}. \tag{7.98}$$

Hence we obtain

$$I(x,\varepsilon) = \int_0^\infty y^{-5/4}\left(G^{m,n+1}_{p+2,q+1}\left|\begin{array}{l} -3/4, (\alpha_p), -1/4 \\ \\ (\beta_q), 1/4\end{array}\right|\frac{1}{t}f\left(\frac{1}{t}\right)\right)(y)$$

$$\times\int_0^\infty G^{q-m+1,p-n}_{p+1,q+1}\left(\frac{x}{u}\left|\begin{array}{l} -(\alpha_p^{n+1}), -(\alpha_n), 3/4 \\ \\ -1/4, -(\beta_q^{m+1}), -(\beta_m)\end{array}\right.\right) e^{-y/u}u^{-7/4-\varepsilon}du dy. \tag{7.99}$$

Now we represent the inner integral in terms of gamma-functions by using the definition of the G-function, Euler's integral (1.22) and Fubini's theorem. As a result we obtain, that

$$I(x,\varepsilon)=\int_0^\infty y^{-2-\varepsilon}\left(G^{m,n+1}_{p+2,q+1}\left|\begin{matrix}-3/4,(\alpha_p),-1/4\\ \\(\beta_q),1/4\end{matrix}\right|\frac{1}{t}f\left(\frac{1}{t}\right)\right)(y)$$

$$\times\frac{1}{2\pi i}\int_{L_\xi}\frac{\Gamma(-1/4+s)\Gamma(3/4+\varepsilon-s)}{\Gamma(3/4+s)\Psi(1-s)}\left(\frac{x}{y}\right)^{-s}dsdy. \tag{7.100}$$

By this expression of $I(x,\varepsilon)$ we can see that it converges uniformly when $\varepsilon\to 0$ owing to the Lebesgue theorem, since the included G-transform belongs to $L_1(\mathbf{R}_+;e^{(2x)^{-1}}x^{-3/2})$. Then

$$I(x)=\lim_{\varepsilon\to+0}I(x,\varepsilon)=\int_0^\infty y^{-2}\left(G^{m,n+1}_{p+2,q+1}\left|\begin{matrix}-3/4,(\alpha_p),-1/4\\ \\(\beta_q),1/4\end{matrix}\right|\frac{1}{t}f\left(\frac{1}{t}\right)\right)(y)$$

$$\times G^{q-m+1,p-n+1}_{p+1,q+1}\left(\frac{x}{y}\left|\begin{matrix}1/4,-(\alpha_p^{n+1}),-(\alpha_n),3/4\\ \\-1/4,-(\beta_q^{m+1}),-(\beta_m)\end{matrix}\right.\right)dy. \tag{7.101}$$

For the G-function in (7.101) one can choose the contour σ as in formula (7.49) due to conditions on its parameters. After that in order to obtain our expansion (7.77) it is sufficient to use the analog of Theorem 7.5. In other words substitute the integral like (1.59) for the respective G-function and change the order of integration by the Fubini theorem. As result we arrive to the integral (7.49), i.e. $I(x)=f(x)$. This completes the proof of Theorem 7.6. •

Let us consider special cases of the general formulae (7.75), (7.77). Setting in (7.75) $m=n=p=0,\ q=1,\ \beta_1=-3/4$ use identity (7.67) and the translation formula (1.62) for the respective G-function. Then up to the simple replacement we arrive to the Lebedev-Skalskaya transform like (6.43)

$$g(\tau)=\frac{1}{\sqrt{\pi}}\int_0^\infty e^{-(2u)^{-1}}\Re K_{1/2+i\tau}\left(\frac{1}{2u}\right)f(u)u^{-5/4}du,\quad \tau>0. \tag{7.102}$$

The corresponding inversion formula (7.77) in this case follows immediately applying (7.83). Hence invoking with (7.68) after changing the contour L_ξ on the right loop and passing to the limit under the sign of integral with additional condition, for instance $g(\tau)\in L(\mathbf{R}_+;\tau\sinh(2\pi\tau))$ we shall obtain

$$f(x)=\frac{2e^{(2x)^{-1}}}{\pi^{3/2}x^{3/4}}\int_0^\infty\cosh(\pi\tau)\Re K_{1/2+i\tau}\left(\frac{1}{2x}\right)g(\tau)d\tau,\quad x>0. \tag{7.103}$$

Another example is given by the $\Re$-transform (7.75), when $m=0,n=p=q=1,\alpha_1=1/4,\beta_1=-1/4$. Appealing to identity (7.66) we obtain the following

$\Re$-transform

$$g(\tau) = \frac{2}{\sqrt{\pi}} \int_0^\infty \Re \left[K_{i\tau}\left(\frac{1}{\sqrt{u}}\right) K_{1+i\tau}\left(\frac{1}{\sqrt{u}}\right) \right] f(u) u^{-5/4} du, \quad \tau > 0. \tag{7.104}$$

Its inversion can be deduced directly by calculation of the Meijer G-function owing to the Table of the Mellin transforms and G-functions in Prudnikov et al. [3]. Omitting here the exact conditions of passing to the limit in the inversion formula (7.77) we demonstrate formally the reciprocal formula of the $\Re$-transform (7.104). In fact, using relation 8.4.21.26 in Prudnikov et al. [3] and identity (7.62) we obtain

$$f(x) = \frac{2x^{-3/4}}{\sqrt{\pi}} \int_0^\infty \Re \left[I_{-i\tau}\left(\frac{1}{\sqrt{x}}\right) I_{-1-i\tau}\left(\frac{1}{\sqrt{x}}\right) \right.$$

$$\left. - I_{i\tau}\left(\frac{1}{\sqrt{x}}\right) I_{1+i\tau}\left(\frac{1}{\sqrt{x}}\right) \right] g(\tau) d\tau, \quad x > 0, \tag{7.105}$$

where $I_\mu(z)$ is the modified Bessel function defined by formula (1.90). More general situation of the index transform with the combination of Bessel functions can be derived by using formula (7.69). In this case we arrive to the following pair of the reciprocal formulae

$$g(\tau) = \frac{2}{\sqrt{\pi}} \int_0^\infty \Re \left[K_{\alpha+i\tau}\left(\frac{1}{\sqrt{u}}\right) K_{\beta+i\tau}\left(\frac{1}{\sqrt{u}}\right) \right] f(u) u^{-5/4} du, \quad \tau > 0, \tag{7.106}$$

$$f(x) = \frac{2x^{-3/4}}{\sqrt{\pi}} \int_0^\infty \Re \left[I_{-\alpha-i\tau}\left(\frac{1}{\sqrt{x}}\right) I_{-\beta-i\tau}\left(\frac{1}{\sqrt{x}}\right) \right.$$

$$\left. - I_{\alpha+i\tau}\left(\frac{1}{\sqrt{x}}\right) I_{\beta+i\tau}\left(\frac{1}{\sqrt{x}}\right) \right] g(\tau) d\tau, \quad x > 0, \tag{7.107}$$

where we mean $\alpha + \beta = 1$. One can exhibit here also the pair of dual formulae with the squares of the Bessel functions, namely

$$g(\tau) = \int_0^\infty \Re \left[J^2_{-1/2-i\tau}(\sqrt{y}) - J^2_{1/2+i\tau}(\sqrt{y}) \right] f(y) dy, \tag{7.108}$$

$$f(x) = \frac{\pi}{2} \int_0^\infty \Re \left[Y^2_{1/2+i\tau}(\sqrt{y}) + J^2_{1/2+i\tau}(\sqrt{y}) \right] g(\tau) d\tau. \tag{7.109}$$

Similarly this approach one can spread on the $\Im$-type index transforms and to obtain a new pairs of theirs.

7.3 Note on the essentially multidimensional Kontorovich-Lebedev transform

We conclude this final chapter with the brief announcement of the idea to spread the above index transform constructions on the multidimensional case. In particular, it concerns the so-called *the essentially multidimensional Kontorovich-Lebedev transform*. We shall introduce it basing on the notions of the multidimensional Fourier transform and certain modification of the essentially multidimensional Laplace transform (see the references at the beginning of this chapter). Note that detail investigation of this object as well as other multidimensional index transforms falls outside of the framework of this book.

As is known the n-dimensional Fourier transform of the function $f : \mathbf{R}^n \to \mathbf{C}$ is defined to be

$$[Ff](x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbf{R}^n} e^{i\langle t,x\rangle} f(t)dt, \tag{7.110}$$

where $x = (x_1, \dots, x_n)$, $\langle t, x\rangle = x_1t_1 + \dots + x_nt_n$. The essentially multidimensional Laplace transform was first introduced in Vu Kim Tuan [6] by formula

$$(\Lambda f)(x) = \int_{\mathbf{R}^n_+} \exp\left(-\max(x_1t_1, \dots, x_nt_n)\right) f(t)dt. \tag{7.111}$$

We need also to define here the multidimensional Mellin transform (see, for example, Brychkov et al. [1,1992]) as follows

$$f^*(s) = \int_{\mathbf{R}^n_+} x^{s-1} f(x)dx, \tag{7.112}$$

where we mean as usually $s \in \mathbf{C}^n$, $f : \mathbf{R}^n_+ \to \mathbf{C}$ and integral (7.112) is understood as

$$f^*(s) = \int_0^\infty x_1^{s_1-1} \int_0^\infty x_2^{s_2-1} \dots \int_0^\infty x_n^{s_n-1} f(x)dx_1 \dots dx_n. \tag{7.113}$$

Let us consider the n-dimensional analog of integral representation (1.98) of the Macdonald function. Namely, we introduce the following kernel

$$K_{i\tau}^{\max}(x) = \frac{1}{2^n} \int_{\mathbf{R}^n} \exp\left(-\max(x_1 \cosh u_1, \dots, x_n \cosh u_n) + i\langle\tau, u\rangle\right) du, \tag{7.114}$$

where $x \in \mathbf{R}^n_+$, $\tau \in \mathbf{R}^n$. Consequently, it gives us the possibility to define the essentially multidimensional Kontorovich-Lebedev transform of the function $f : \mathbf{R}^n \to \mathbf{C}$ accordingly

$$K_{i\tau}^{\max}[f] = \int_{\mathbf{R}^n_+} K_{i\tau}^{\max}(x) f(x)dx. \tag{7.115}$$

As is obvious, the case $n = 1$ leads us to the Kontorovich-Lebedev transform (2.1). However, one can express the introduced kernel (7.114) for the Kontorovich-Lebedev transform (7.115) in terms of the special functions of several variables. For this we exhibit the key Mellin transform formula of the Laplace kernel $\exp(-\max(x_1, \dots, x_n))$ (see Brychkov et al. [1,1992])

$$\int_{\mathbf{R}^n_+} x^{s-1} \exp(-\max(x_1, \dots, x_n))dx = \frac{\Gamma(1 + s_1 + \dots + s_n)}{s_1 \dots s_n}. \tag{7.116}$$

It immediately implies that the inversion formula is true

$$\exp(-\max(x_1,\dots,x_n)) = \frac{1}{(2\pi i)^n}\int_{(\nu_1)}\cdots\int_{(\nu_n)}\frac{\Gamma(1+s_1+\ldots+s_n)}{s_1\ldots s_n}$$

$$\times x_1^{-s_1}\ldots x_n^{-s_n}ds_1\ldots ds_n, \tag{7.117}$$

where $\nu_i = \Re s_i > 0$, $i = 1,\dots,n$. Making use this representation substitute it within (7.114) and change the order of integration. To evaluate the obtained inner integrals invoke with integral (1.104) namely with its reciprocal relation as the inverse cosine Fourier transform (see also formula 2.5.46.6 in Prudnikov et al. [1]). Calling the reduction formula (1.23) for gamma-functions as a result we find that

$$K_{i\tau}^{\max}(x) = \frac{1}{(2\pi i)^n}\int_{(\nu_1)}\cdots\int_{(\nu_n)}\frac{\Gamma(1+s_1+\ldots+s_n)}{s_1\ldots s_n}x_1^{-s_1}\ldots x_n^{-s_n}ds_1\ldots ds_n$$

$$\times\prod_{j=1}^{n}\int_0^\infty\frac{\cos\tau_j u}{\cosh^{s_j}u}du = \frac{4^{-n}}{(2\pi i)^n}\int_{(\nu_1)}\cdots\int_{(\nu_n)}\frac{\Gamma(1+s_1+\ldots+s_n)}{\Gamma(s_1+1)\ldots\Gamma(s_n+1)}$$

$$\times\prod_{j=1}^{n}\Gamma\left(\frac{s_j+i\tau_j}{2}\right)\Gamma\left(\frac{s_j-i\tau_j}{2}\right)\left(\frac{x_j}{2}\right)^{-s_j}ds_j. \tag{7.118}$$

As is shown for example in Marichev and Vu Kim Tuan [1] integral (7.118) is easy to reduce to the H-function of several variables. Furthermore, applying the Mellin-Parseval formula for the multidimensional Mellin transform (7.112) (see Brychkov et al. [1,1992]) one can deduce the relation for the Kontorovich-Lebedev transform (7.115) such that

$$K_{i\tau}^{\max}[f] = \frac{4^{-n}}{(2\pi i)^n}\int_{(\nu_1)}\cdots\int_{(\nu_n)}\frac{\Gamma(1+s_1+\ldots+s_n)}{\Gamma(s_1+1)\ldots\Gamma(s_n+1)}$$

$$\times\prod_{j=1}^{n}\Gamma\left(\frac{s_j+i\tau_j}{2}\right)\Gamma\left(\frac{s_j-i\tau_j}{2}\right)2^{s_j}f^*(1-s)ds, \tag{7.119}$$

under condition $f \in L_1(\mathbf{R}_+^n; x^{-\nu})$, $\nu = (\nu_1,\dots,\nu_n) \in \mathbf{R}_+^n$. Here we mean that $s = (s_1,\dots,s_n)$, $ds = ds_1\ldots ds_n$.

Let us demonstrate the formal deduction of the inversion formula for the multidimensional Kontorovich-Lebedev transform (7.115). For this use identity (7.34) to organize n-times integration through in (7.119) by indices $\tau_j, j = 1,\dots,n$. Change formally the order of integration after calculation of the inner integrals we arrive to the equality

$$\left(\frac{2}{\pi^2}\right)^n\int_{\mathbf{R}_+^n}\prod_{j=1}^{n}\tau_j\sinh(\pi\tau_j)K_{i\tau_j}(x_j)K_{i\tau}^{\max}[f]d\tau$$

$$= \frac{1}{(2\pi i)^n}\int_{(\nu_1)}\cdots\int_{(\nu_n)}\frac{\Gamma(1+s_1+\ldots+s_n)}{\Gamma(s_1+1)\ldots\Gamma(s_n+1)}x^s f^*(1-s)ds. \tag{7.120}$$

Choose now the contour $(\nu) = (\nu_1 - i\infty, \nu_1 + i\infty) \times \ldots \times (\nu_n - i\infty, \nu_n + i\infty)$ with $0 < \nu_j < 1, j = 1, \ldots, n$ and $\sum \nu_j > n - 1$. Consider the auxiliary kernel that can be easily reduced to the Meijer G-function of several variables, namely

$$\mathcal{G}(x) = \frac{1}{(2\pi i)^n} \int_{(\nu)} \frac{\Gamma(s_1) \ldots \Gamma(s_n)}{\Gamma(s_1 + \ldots + s_n + 1 - n)} x^{-s} ds. \tag{7.121}$$

This function generates the kernel of the inverse Kontorovich-Lebedev transform. Indeed, let us define the kernel such that

$$K_{i\tau}^{\max^{-1}}(x) = \int_{\mathbf{R}_+^n} \mathcal{G}(x \circ y) \prod_{j=1}^{n} K_{i\tau_j}(y_j) dy, \tag{7.122}$$

where we mean by $x \circ y = (x_1 y_1, \ldots, x_n y_n)$. Hence applying through in (7.120) this operator of the Mellin convolution type change formally the order of integration and we obtain

$$\left(\frac{2}{\pi^2}\right)^n \int_{\mathbf{R}_+^n} \prod_{j=1}^{n} \tau_j \sinh(\pi \tau_j) K_{i\tau}^{\max^{-1}}(x) K_{i\tau}^{\max}[f] d\tau$$
$$= \frac{1}{(2\pi i)^n} \int_{(\nu)} x^{-s-1} f^*(1 - s) ds = \frac{f(x^{-1})}{x_1^2 \ldots x_n^2}, \tag{7.123}$$

where $x^{-1} = (x_1^{-1}, \ldots, x_n^{-1})$. Thus we deduced the pair of reciprocal formulae of the essentially multidimensional Kontorovich-Lebedev transform

$$K_{i\tau}^{\max}[f] = \int_{\mathbf{R}_+^n} K_{i\tau}^{\max}(x) f(x) dx,$$

$$\frac{f(x^{-1})}{x_1^2 \ldots x_n^2} = \left(\frac{2}{\pi^2}\right)^n \int_{\mathbf{R}_+^n} \prod_{j=1}^{n} \tau_j \sinh(\pi \tau_j) K_{i\tau}^{\max^{-1}}(x) K_{i\tau}^{\max}[f] d\tau. \tag{7.124}$$

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Notations